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that the function $h(x) = f(g(x))$ is differentiable with derivative $h'(x) = f'(g(x))\,g'(x)$. For any non-zero infinitesimal dx, write $dg = g(x+dx) - g(x)$ and $dh = h(x+dx) - h(x)$ then

$$dh = f(g(x+dx)) - f(g(x)) = f(g(x) + dg) - f(g(x)).$$

We want to show that for any non-zero infinitesimal dx,

$$\frac{dh}{dx} \approx f'(g(x))\,g'(x)\,. \tag{1}$$

Let non-zero infinitesimal dx be given. By continuity of $g(x)$, dg is also infinitesimal.

Case 1. $dg = 0$. Then $dh = 0$, so ${}^{\circ}\!\left(\frac{dg}{dx}\right) = g'(x) = 0$ and $\frac{dh}{dx} = 0$. Thus both sides of (1) are zero, so (1) holds.

Case 2. $dg \neq 0$. Then $\frac{dh}{dx} = \frac{dh}{dg} \cdot \frac{dg}{dx}$ that is

$$\frac{dh}{dx} = \frac{f(g(x)+dg) - f(g(x))}{dg} \cdot \frac{g(x+dx) - g(x)}{dx}\,. \tag{2}$$

The two factors of the right side of (2) are infinitely close to $f'(g(x))$ and $g'(x)$ respectively. Now using the rules given in Section 2 for manipulating the symbol $\approx$ we get

$$\frac{dh}{dx} \approx f'(g(x)) \cdot g'(x)$$

as desired.

10. Integration

Let $f(x)$ be a standard function integrable on the standard interval $[a, b]$. For each standard n let

$$a = a_0^n < a_1^n < \cdots < a_n^n = b$$

be a partition of the interval into n subintervals of equal length. The Riemann sums

$$S_n = \sum_{i=1}^{n} f(a_i^n)\,(a_i^n - a_{i-1}^n)$$

constitute an infinite sequence, and by the Main Theorem this sequence can be extended to a sequence defined on N^*. For an infinite natural number α it seems natural to denote the α^{th} term S_α by

$$\sum_{i=1}^{\alpha} f(a_i^\alpha)\,(a_i^\alpha - a_{i-1}^\alpha)\,. \tag{1}$$

We might think of this as a "Riemann sum" on an infinitely fine net. The use of $\sum$ notation seems appropriate because the "sum" shares (by virtue of the Main Theorem) many properties of standard finite sums. For example, the property (omitting the summands for brevity)

$$\sum_{i=1}^{\alpha} = \sum_{i=1}^{\beta} + \sum_{i=\beta+1}^{\alpha}\,.$$

Now since

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(a_i^n)\,(a_i^n - a_{i-1}^n) = \int_a^b f(x)\,dx\,,$$

we see, using the non-standard characterization of the notion of limit of a sequence, that if α is an infinite natural number

$$\sum_{i=1}^{\alpha} f(a_i^\alpha)\,(a_i^\alpha - a_{i-1}^\alpha) \approx \int_a^b f(x)\,dx\,.$$

A further development of the theory of Integration and in particular a non-standard characterization of the Riemann integrable functions requires more machinery than we are prepared to set up here.

11. The Main Theorem Revisited

The version of the Main Theorem which we gave you in Section 3 is a specialization of a considerably more general result. While we stated it in terms of the real number system R, it happens to be true of any non-empty set X whatsoever. This opens the way for a penetration of the methods of Non-Standard Analysis to other branches of mathematics. For example, one might extend the complex number system C to a field C^*. There one could have "polygons" with sides of infinitely small length and vertices indexed by the initial segment of N^* determined by some infinite natural number.