

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 20 (1974)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: NON-STANDARD ANALYSIS: AN EXPOSITION
Autor: Levitz, Hilbert
Kapitel: 10. Integration
DOI: <https://doi.org/10.5169/seals-46892>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 07.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

that the function $h(x) = f(g(x))$ is differentiable with derivative $h'(x) = f'(g(x))g'(x)$. For any non-zero infinitesimal dx , write $dg = g(x+dx) - g(x)$ and $dh = h(x+dx) - h(x)$ then

$$dh = f(g(x+dx)) - f(g(x)) = f(g(x) + dg) - f(g(x)).$$

We want to show that for any non-zero infinitesimal dx ,

$$(1) \quad \frac{dh}{dx} \approx f'(g(x))g'(x).$$

Let non-zero infinitesimal dx be given. By continuity of $g(x)$, dg is also infinitesimal.

Case 1. $dg = 0$. Then $dh = 0$, so $\circ(\frac{dg}{dx}) = g'(x) = 0$ and $\frac{dh}{dx} = 0$. Thus both sides of (1) are zero, so (1) holds.

Case 2. $dg \neq 0$. Then $\frac{dh}{dx} = \frac{dh}{dg} \cdot \frac{dg}{dx}$ that is

$$(2) \quad \frac{dh}{dx} = \frac{f(g(x)+dg) - f(g(x))}{dg} \cdot \frac{g(x+dx) - g(x)}{dx}.$$

The two factors of the right side of (2) are infinitely close to $f'(g(x))$ and $g'(x)$ respectively. Now using the rules given in Section 2 for manipulating the symbol \approx we get

$$\frac{dh}{dx} \approx f'(g(x)) \cdot g'(x)$$

as desired.

10. INTEGRATION

Let $f(x)$ be a standard function integrable on the standard interval $[a, b]$. For each standard n let

$$a = a_0^n < a_1^n < \cdots < a_n^n = b$$

be a partition of the interval into n subintervals of equal length. The Riemann sums

$$S_n = \sum_{i=1}^n f(a_i^n)(a_i^n - a_{i-1}^n)$$

constitute an infinite sequence, and by the Main Theorem this sequence can be extended to a sequence defined on N^* . For an infinite natural number α it seems natural to denote the α^{th} term S_α by

$$(1) \quad \sum_{i=1}^{\alpha} f(a_i^\alpha) (a_i^\alpha - a_{i-1}^\alpha).$$

We might think of this as a “Riemann sum” on an infinitely fine net. The use of \sum notation seems appropriate because the “sum” shares (by virtue of the Main Theorem) many properties of standard finite sums. For example, the property (omitting the summands for brevity)

$$\sum_{i=1}^{\alpha} = \sum_{i=1}^{\beta} + \sum_{i=\beta+1}^{\alpha}.$$

Now since

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(a_i^n) (a_i^n - a_{i-1}^n) = \int_a^b f(x) dx,$$

we see, using the non-standard characterization of the notion of limit of a sequence, that if α is an infinite natural number

$$\sum_{i=1}^{\alpha} f(a_i^\alpha) (a_i^\alpha - a_{i-1}^\alpha) \approx \int_a^b f(x) dx.$$

A further development of the theory of Integration and in particular a non-standard characterization of the Riemann integrable functions requires more machinery than we are prepared to set up here.

11. THE MAIN THEOREM REVISITED

The version of the Main Theorem which we gave you in Section 3 is a specialization of a considerably more general result. While we stated it in terms of the real number system R , it happens to be true of any non-empty set X whatsoever. This opens the way for a penetration of the methods of Non-Standard Analysis to other branches of mathematics. For example, one might extend the complex number system C to a field C^* . There one could have “polygons” with sides of infinitely small length and vertices indexed by the initial segment of N^* determined by some infinite natural number.