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that the function  $h(x) = f(g(x))$  is differentiable with derivative  $h'(x) = f'(g(x))g'(x)$ . For any non-zero infinitesimal  $dx$ , write  $dg = g(x+dx) - g(x)$  and  $dh = h(x+dx) - h(x)$  then

$$dh = f(g(x+dx)) - f(g(x)) = f(g(x) + dg) - f(g(x)).$$

We want to show that for any non-zero infinitesimal  $dx$ ,

$$(1) \quad \frac{dh}{dx} \approx f'(g(x))g'(x).$$

Let non-zero infinitesimal  $dx$  be given. By continuity of  $g(x)$ ,  $dg$  is also infinitesimal.

Case 1.  $dg = 0$ . Then  $dh = 0$ , so  $\left(\frac{dg}{dx}\right) = g'(x) = 0$  and  $\frac{dh}{dx} = 0$ . Thus

both sides of (1) are zero, so (1) holds.

Case 2.  $dg \neq 0$ . Then  $\frac{dh}{dx} = \frac{dh}{dg} \cdot \frac{dg}{dx}$  that is

$$(2) \quad \frac{dh}{dx} = \frac{f(g(x)+dg) - f(g(x))}{dg} \cdot \frac{g(x+dx) - g(x)}{dx}.$$

The two factors of the right side of (2) are infinitely close to  $f'(g(x))$  and  $g'(x)$  respectively. Now using the rules given in Section 2 for manipulating the symbol  $\approx$  we get

$$\frac{dh}{dx} \approx f'(g(x)) \cdot g'(x)$$

as desired.

## 10. INTEGRATION

Let  $f(x)$  be a standard function integrable on the standard interval  $[a, b]$ . For each standard  $n$  let

$$a = a_0^n < a_1^n < \cdots < a_n^n = b$$

be a partition of the interval into  $n$  subintervals of equal length. The Riemann sums

$$S_n = \sum_{i=1}^n f(a_i^n) (a_i^n - a_{i-1}^n)$$

constitute an infinite sequence, and by the Main Theorem this sequence can be extended to a sequence defined on  $N^*$ . For an infinite natural number  $\alpha$  it seems natural to denote the  $\alpha^{\text{th}}$  term  $S_\alpha$  by

$$(1) \quad \sum_{i=1}^{\alpha} f(a_i^\alpha) (a_i^\alpha - a_{i-1}^\alpha).$$

We might think of this as a “Riemann sum” on an infinitely fine net. The use of  $\sum$  notation seems appropriate because the “sum” shares (by virtue of the Main Theorem) many properties of standard finite sums. For example, the property (omitting the summands for brevity)

$$\sum_{i=1}^{\alpha} = \sum_{i=1}^{\beta} + \sum_{i=\beta+1}^{\alpha}.$$

Now since

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(a_i^n) (a_i^n - a_{i-1}^n) = \int_a^b f(x) dx,$$

we see, using the non-standard characterization of the notion of limit of a sequence, that if  $\alpha$  is an infinite natural number

$$\sum_{i=1}^{\alpha} f(a_i^\alpha) (a_i^\alpha - a_{i-1}^\alpha) \approx \int_a^b f(x) dx.$$

A further development of the theory of Integration and in particular a non-standard characterization of the Riemann integrable functions requires more machinery than we are prepared to set up here.

## 11. THE MAIN THEOREM REVISITED

The version of the Main Theorem which we gave you in Section 3 is a specialization of a considerably more general result. While we stated it in terms of the real number system  $R$ , it happens to be true of any non-empty set  $X$  whatsoever. This opens the way for a penetration of the methods of Non-Standard Analysis to other branches of mathematics. For example, one might extend the complex number system  $C$  to a field  $C^*$ . There one could have “polygons” with sides of infinitely small length and vertices indexed by the initial segment of  $N^*$  determined by some infinite natural number.