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f(x) = x is a continuous function which is infinite at  $x = x_0$ . On the other hand, if  $x_0$  were finite then  $x_0$  is infinitely close to some standard number  $y_0$ , and since (7) fails  $y_0 \notin S$ . Thus the function  $f(x) = \frac{1}{x - y_0}$  is continuous on S (the denominator can't be zero for  $x \in S$  because  $y_0 \notin S$ ); moreover,  $f(x_0)$  is infinite since  $x_0 - y_0$  is a non-zero infinitesimal.

It is known in the study of the topology of the real line that a necessary and sufficient condition for a set S to have the property that all continuous functions on it be bounded is that S be compact. We've just shown that (7) is also necessary and sufficient, so this establishes the following theorems.

THEOREM 6.5. A set  $S \subseteq R$  is compact if and only if every point of  $S^*$  is infinitely close to a point of S.

THEOREM 6.6 If the standard function f(x) is continuous on a standard compact set S, then f(x) is bounded there.

It turns out that in applying the methods of Non-standard Analysis to the subject of General Topology, the characterization of compactness given by Theorem 6.5 still holds.

The theorem below gives a very nice characterization of the notion of a uniformly continuous function. We shall not deny you the pleasure of trying to prove it yourself. The proof of Theorem 6.1 should provide the inspiration.

THEOREM 6.7. A standard function is uniformly continuous on the standard set S if and only if  $x \approx y$  implies  $f^*(x) \approx f^*(y)$  for all  $x, y \in S^*$ . Using the above theorem we can quickly dispatch the following.

THEOREM 6.8. A standard function f continuous on a compact standard set S is uniformly continuous on S.

PROOF. Let  $x, y \in S^*$  be given such that  $x \approx y$ . By compactness of S there exists  $x_0 \in S$  such that  $x \approx x_0$ . Since  $\approx$  is an equivalence relation  $x \approx x_0 \approx y$ . Now by continuity  $f^*(x) \approx f(x_0) \approx f^*(y)$ , therefore  $f^*(x) \approx f^*(y)$ .

# 7. Infinite Sequences

An infinite sequence  $\{a_n\}$  can be thought of as a function from N into R. Accordingly the Main Theorem provides for an extension function from  $N^*$  into  $R^*$ . Put differently, after we exhaust all the terms with finite

subscripts, the sequence continues on with infinite subscripts as follows:

$$\underbrace{a_1, a_2, ..., a_n ...}_{\text{terms with}} \qquad \underbrace{... a_{\alpha-1}, a_{\alpha}, a_{\alpha+1} ...}_{\text{terms with}}$$

$$\text{finite} \qquad \text{infinite}$$

$$\text{subscripts} \qquad \text{subscripts}$$

It is easy to see that the sequence

continues to have the value 0 when we look at its extension because the statement

$$(\forall x) (x \in N \rightarrow a_x = 0)$$

is true in R and therefore in  $R^*$ . Likewise the sequence

continues to alternate, and the sequence of primes  $p_1, p_2, p_3, ..., p_n, ...$  when extended "enumerates" the primes of  $N^*$ .

Various properties of standard sequences can be characterized in terms of what happens to the terms with infinite subscripts (intuitively—when you get out to infinity).

In what follows  $\{a_n\}$ ,  $\{b_n\}$  will be standard sequences and a, b will be standard numbers. The proof of the following theorem runs along lines which by now should be familiar to you.

### THEOREM 7.1.

- (i)  $\{a_n\}$  is bounded iff  $a_{\alpha}$  is finite for all infinite natural numbers  $\alpha$ .
- (ii)  $\lim_{n\to\infty} a_n = a$  iff  $a_{\alpha} \approx a$  for all infinite natural numbers  $\alpha$ .
- (iii)  $\lim_{n\to\infty} a_n = \infty$  iff  $a_{\alpha}$  is infinite for all infinite natural numbers  $\alpha$ .
- (iv)  $\{a_n\}$  is a Cauchy sequence iff  $a_{\alpha} \approx a_{\beta}$  for all infinite natural numbers  $\alpha$ ,  $\beta$ .

Example 7.1. Suppose

$$\lim_{n\to\infty} a_n = a \text{ and } \lim_{n\to\infty} b_n = b,$$

and we want to show

$$\lim_{n\to\infty} (a_n + b_n) = a + b \text{ and } \lim_{n\to\infty} a_n b_n = a b.$$

Let  $\alpha$  be an infinite natural number. By the above theorem we have  $a_{\alpha} \approx a$  and  $b_{\alpha} \approx b$ . From this we see easily that  $a_{\alpha}$  and  $b_{\alpha}$  are finite. Now using the rules given in Section 2 for manipulating the  $\approx$  symbol,

$$a_{\alpha} + b_{\alpha} \approx a + b$$
 and  $a_{\alpha} b_{\alpha} \approx a b$ .

Thus by the above theorem, the desired results are established.

Example 7.2. Suppose we wanted to calculate

$$\lim_{n\to\infty} (n^2 - n) = ?$$

We can proceed directly—let  $\alpha$  be an arbitrary infinite natural number, then

$$\alpha^2 - \alpha = \alpha (\alpha - 1) = \text{(infinite) (infinite)}$$
  
= infinite

thus

$$\lim_{n\to\infty} (n^2 - n) = \infty.$$

## 8. Infinitely Fine Partitions of an Interval

Consider the familiar process of partitioning an interval [a, b] into n subintervals of equal length by means of the partition points

$$a = a_0 < a_1 < \cdots < a_n = b.$$

If we let  $a_i^j$  denote the  $i^{th}$  partition point when the interval is divided into j subintervals of equal length, it is easily seen that

$$a_i^j = a + \left(\frac{b-a}{i}\right)i.$$

Now the right side of this expression is a function from  $I \times I$  into R, where  $I \subseteq R$  is the set of integers. By the Main Theorem this function extends to a function from  $I^* \times I^*$  into  $R^*$ . We continue to use  $a_i^j$  for the image under this extended function. If we let  $\alpha$  be a fixed infinite natural number, then for  $0 \le i \le \alpha$ ,  $a_i^{\alpha}$  must lie in the interval  $[a, b]^*$ . Note that the  $i^{th}$  sub-

interval  $[a_i^{\alpha}, a_{i+1}^{\alpha}]$  has the infinitesimal  $\frac{b-a}{\alpha}$  as its length. Two such intervals

can intersect only if they have an end point in common, and the intersection is that end point. Each partition point  $a_i^{\alpha}$  (other than a, b) has an immediately