

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 20 (1974)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: NON-STANDARD ANALYSIS: AN EXPOSITION
Autor: Levitz, Hilbert
Kapitel: 7. Infinité Sequences
DOI: <https://doi.org/10.5169/seals-46892>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 18.02.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

$f(x) = x$ is a continuous function which is infinite at $x = x_0$. On the other hand, if x_0 were finite then x_0 is infinitely close to some standard number y_0 , and since (7) fails $y_0 \notin S$. Thus the function $f(x) = \frac{1}{x-y_0}$ is continuous on S (the denominator can't be zero for $x \in S$ because $y_0 \notin S$); moreover, $f(x_0)$ is infinite since $x_0 - y_0$ is a non-zero infinitesimal.

It is known in the study of the topology of the real line that a necessary and sufficient condition for a set S to have the property that all continuous functions on it be bounded is that S be compact. We've just shown that (7) is also necessary and sufficient, so this establishes the following theorems.

THEOREM 6.5. A set $S \subseteq R$ is compact if and only if every point of S^* is infinitely close to a point of S .

THEOREM 6.6 If the standard function $f(x)$ is continuous on a standard compact set S , then $f(x)$ is bounded there.

It turns out that in applying the methods of Non-standard Analysis to the subject of General Topology, the characterization of compactness given by Theorem 6.5 still holds.

The theorem below gives a very nice characterization of the notion of a uniformly continuous function. We shall not deny you the pleasure of trying to prove it yourself. The proof of Theorem 6.1 should provide the inspiration.

THEOREM 6.7. A standard function is uniformly continuous on the standard set S if and only if $x \approx y$ implies $f^*(x) \approx f^*(y)$ for all $x, y \in S^*$.

Using the above theorem we can quickly dispatch the following.

THEOREM 6.8. A standard function f continuous on a compact standard set S is uniformly continuous on S .

PROOF. Let $x, y \in S^*$ be given such that $x \approx y$. By compactness of S there exists $x_0 \in S$ such that $x \approx x_0$. Since \approx is an equivalence relation $x \approx x_0 \approx y$. Now by continuity $f^*(x) \approx f(x_0) \approx f^*(y)$, therefore $f^*(x) \approx f^*(y)$.

7. INFINITE SEQUENCES

An infinite sequence $\{a_n\}$ can be thought of as a function from N into R . Accordingly the Main Theorem provides for an extension function from N^* into R^* . Put differently, after we exhaust all the terms with finite

subscripts, the sequence continues on with infinite subscripts as follows:

$$\begin{array}{ccc} \overbrace{a_1, a_2, \dots, a_n \dots} & & \overbrace{\dots a_{\alpha-1}, a_{\alpha}, a_{\alpha+1} \dots} \\ \text{terms with} & & \text{terms with} \\ \text{finite} & & \text{infinite} \\ \text{subscripts} & & \text{subscripts} \end{array}$$

It is easy to see that the sequence

$$0, 0, \dots, 0 \dots$$

continues to have the value 0 when we look at its extension because the statement

$$(\forall x) (x \in N \rightarrow a_x = 0)$$

is true in R and therefore in R^* . Likewise the sequence

$$1, 0, 1, 0, \dots, 1, 0, \dots$$

continues to alternate, and the sequence of primes $p_1, p_2, p_3, \dots, p_n, \dots$ when extended “enumerates” the primes of N^* .

Various properties of standard sequences can be characterized in terms of what happens to the terms with infinite subscripts (intuitively—when you get out to infinity).

In what follows $\{a_n\}$, $\{b_n\}$ will be standard sequences and a, b will be standard numbers. The proof of the following theorem runs along lines which by now should be familiar to you.

THEOREM 7.1.

- (i) $\{a_n\}$ is bounded iff a_{α} is finite for all infinite natural numbers α .
- (ii) $\lim_{n \rightarrow \infty} a_n = a$ iff $a_{\alpha} \approx a$ for all infinite natural numbers α .
- (iii) $\lim_{n \rightarrow \infty} a_n = \infty$ iff a_{α} is infinite for all infinite natural numbers α .
- (iv) $\{a_n\}$ is a Cauchy sequence iff $a_{\alpha} \approx a_{\beta}$ for all infinite natural numbers α, β .

Example 7.1. Suppose

$$\lim_{n \rightarrow \infty} a_n = a \text{ and } \lim_{n \rightarrow \infty} b_n = b,$$

and we want to show

$$\lim_{n \rightarrow \infty} (a_n + b_n) = a + b \text{ and } \lim_{n \rightarrow \infty} a_n b_n = a b.$$

Let α be an infinite natural number. By the above theorem we have $a_\alpha \approx a$ and $b_\alpha \approx b$. From this we see easily that a_α and b_α are finite. Now using the rules given in Section 2 for manipulating the \approx symbol,

$$a_\alpha + b_\alpha \approx a + b \text{ and } a_\alpha b_\alpha \approx a b.$$

Thus by the above theorem, the desired results are established.

Example 7.2. Suppose we wanted to calculate

$$\lim_{n \rightarrow \infty} (n^2 - n) = ?$$

We can proceed directly— let α be an arbitrary infinite natural number, then

$$\begin{aligned} \alpha^2 - \alpha &= \alpha(\alpha - 1) = (\text{infinite})(\text{infinite}) \\ &= \text{infinite} \end{aligned}$$

thus

$$\lim_{n \rightarrow \infty} (n^2 - n) = \infty.$$

8. INFINITELY FINE PARTITIONS OF AN INTERVAL

Consider the familiar process of partitioning an interval $[a, b]$ into n subintervals of equal length by means of the partition points

$$a = a_0 < a_1 < \dots < a_n = b.$$

If we let a_i^j denote the i^{th} partition point when the interval is divided into j subintervals of equal length, it is easily seen that

$$a_i^j = a + \left(\frac{b-a}{j} \right) i.$$

Now the right side of this expression is a function from $I \times I$ into R , where $I \subseteq R$ is the set of integers. By the Main Theorem this function extends to a function from $I^* \times I^*$ into R^* . We continue to use a_i^j for the image under this extended function. If we let α be a fixed infinite natural number, then for $0 \leq i \leq \alpha$, a_i^α must lie in the interval $[a, b]^*$. Note that the i^{th} sub-

interval $[a_i^\alpha, a_{i+1}^\alpha]$ has the infinitesimal $\frac{b-a}{\alpha}$ as its length. Two such intervals

can intersect only if they have an end point in common, and the intersection is that end point. Each partition point a_i^α (other than a, b) has an immediately