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$f(x) = x$  is a continuous function which is infinite at  $x = x_0$ . On the other hand, if  $x_0$  were finite then  $x_0$  is infinitely close to some standard number  $y_0$ , and since (7) fails  $y_0 \notin S$ . Thus the function  $f(x) = \frac{1}{x-y_0}$  is continuous on  $S$  (the denominator can't be zero for  $x \in S$  because  $y_0 \notin S$ ); moreover,  $f(x_0)$  is infinite since  $x_0 - y_0$  is a non-zero infinitesimal.

It is known in the study of the topology of the real line that a necessary and sufficient condition for a set  $S$  to have the property that all continuous functions on it be bounded is that  $S$  be compact. We've just shown that (7) is also necessary and sufficient, so this establishes the following theorems.

**THEOREM 6.5.** A set  $S \subseteq R$  is compact if and only if every point of  $S^*$  is infinitely close to a point of  $S$ .

**THEOREM 6.6** If the standard function  $f(x)$  is continuous on a standard compact set  $S$ , then  $f(x)$  is bounded there.

It turns out that in applying the methods of Non-standard Analysis to the subject of General Topology, the characterization of compactness given by Theorem 6.5 still holds.

The theorem below gives a very nice characterization of the notion of a uniformly continuous function. We shall not deny you the pleasure of trying to prove it yourself. The proof of Theorem 6.1 should provide the inspiration.

**THEOREM 6.7.** A standard function is uniformly continuous on the standard set  $S$  if and only if  $x \approx y$  implies  $f^*(x) \approx f^*(y)$  for all  $x, y \in S^*$ .

Using the above theorem we can quickly dispatch the following.

**THEOREM 6.8.** A standard function  $f$  continuous on a compact standard set  $S$  is uniformly continuous on  $S$ .

**PROOF.** Let  $x, y \in S^*$  be given such that  $x \approx y$ . By compactness of  $S$  there exists  $x_0 \in S$  such that  $x \approx x_0$ . Since  $\approx$  is an equivalence relation  $x \approx x_0 \approx y$ . Now by continuity  $f^*(x) \approx f^*(x_0) \approx f^*(y)$ , therefore  $f^*(x) \approx f^*(y)$ .

## 7. INFINITE SEQUENCES

An infinite sequence  $\{a_n\}$  can be thought of as a function from  $N$  into  $R$ . Accordingly the Main Theorem provides for an extension function from  $N^*$  into  $R^*$ . Put differently, after we exhaust all the terms with finite

subscripts, the sequence continues on with infinite subscripts as follows:

$$\begin{array}{cc} \underbrace{a_1, a_2, \dots, a_n \dots}_{\text{terms with}} & \underbrace{\dots a_{\alpha-1}, a_{\alpha}, a_{\alpha+1} \dots}_{\text{terms with}} \\ \text{finite} & \text{infinite} \\ \text{subscripts} & \text{subscripts} \end{array}$$

It is easy to see that the sequence

$$0, 0, \dots, 0 \dots$$

continues to have the value 0 when we look at its extension because the statement

$$(\forall x) (x \in N \rightarrow a_x = 0)$$

is true in  $R$  and therefore in  $R^*$ . Likewise the sequence

$$1, 0, 1, 0, \dots, 1, 0, \dots$$

continues to alternate, and the sequence of primes  $p_1, p_2, p_3, \dots, p_n, \dots$  when extended “enumerates” the primes of  $N^*$ .

Various properties of standard sequences can be characterized in terms of what happens to the terms with infinite subscripts (intuitively—when you get out to infinity).

In what follows  $\{a_n\}$ ,  $\{b_n\}$  will be standard sequences and  $a, b$  will be standard numbers. The proof of the following theorem runs along lines which by now should be familiar to you.

**THEOREM 7.1.**

- (i)  $\{a_n\}$  is bounded iff  $a_\alpha$  is finite for all infinite natural numbers  $\alpha$ .
- (ii)  $\lim_{n \rightarrow \infty} a_n = a$  iff  $a_\alpha \approx a$  for all infinite natural numbers  $\alpha$ .
- (iii)  $\lim_{n \rightarrow \infty} a_n = \infty$  iff  $a_\alpha$  is infinite for all infinite natural numbers  $\alpha$ .
- (iv)  $\{a_n\}$  is a Cauchy sequence iff  $a_\alpha \approx a_\beta$  for all infinite natural numbers  $\alpha, \beta$ .

**Example 7.1.** Suppose

$$\lim_{n \rightarrow \infty} a_n = a \text{ and } \lim_{n \rightarrow \infty} b_n = b,$$

and we want to show

$$\lim_{n \rightarrow \infty} (a_n + b_n) = a + b \text{ and } \lim_{n \rightarrow \infty} a_n b_n = a b.$$

Let  $\alpha$  be an infinite natural number. By the above theorem we have  $a_\alpha \approx a$  and  $b_\alpha \approx b$ . From this we see easily that  $a_\alpha$  and  $b_\alpha$  are finite. Now using the rules given in Section 2 for manipulating the  $\approx$  symbol,

$$a_\alpha + b_\alpha \approx a + b \text{ and } a_\alpha b_\alpha \approx a b.$$

Thus by the above theorem, the desired results are established.

Example 7.2. Suppose we wanted to calculate

$$\lim_{n \rightarrow \infty} (n^2 - n) = ?$$

We can proceed directly— let  $\alpha$  be an arbitrary infinite natural number, then

$$\begin{aligned} \alpha^2 - \alpha &= \alpha(\alpha - 1) = (\text{infinite})(\text{infinite}) \\ &= \text{infinite} \end{aligned}$$

thus

$$\lim_{n \rightarrow \infty} (n^2 - n) = \infty.$$

## 8. INFINITELY FINE PARTITIONS OF AN INTERVAL

Consider the familiar process of partitioning an interval  $[a, b]$  into  $n$  subintervals of equal length by means of the partition points

$$a = a_0 < a_1 < \cdots < a_n = b.$$

If we let  $a_i^j$  denote the  $i^{\text{th}}$  partition point when the interval is divided into  $j$  subintervals of equal length, it is easily seen that

$$a_i^j = a + \left( \frac{b-a}{j} \right) i.$$

Now the right side of this expression is a function from  $I \times I$  into  $R$ , where  $I \subseteq R$  is the set of integers. By the Main Theorem this function extends to a function from  $I^* \times I^*$  into  $R^*$ . We continue to use  $a_i^j$  for the image under this extended function. If we let  $\alpha$  be a fixed infinite natural number, then for  $0 \leq i \leq \alpha$ ,  $a_i^\alpha$  must lie in the interval  $[a, b]^*$ . Note that the  $i^{\text{th}}$  sub-

interval  $[a_i^\alpha, a_{i+1}^\alpha]$  has the infinitesimal  $\frac{b-a}{\alpha}$  as its length. Two such intervals

can intersect only if they have an end point in common, and the intersection is that end point. Each partition point  $a_i^\alpha$  (other than  $a, b$ ) has an immediately