$f(x) = x$ is a continuous function which is infinite at $x = x_0$. On the other hand, if x_0 were finite then x_0 is infinitely close to some standard number y_0, and since (7) fails $y_0 \notin S$. Thus the function $f(x) = \dfrac{1}{x—y_0}$ is continuous on S (the denominator can't be zero for $x \in S$ because $y_0 \notin S$); moreover, $f(x_0)$ is infinite since $x_0 - y_0$ is a non-zero infinitesimal.

It is known in the study of the topology of the real line that a necessary and sufficient condition for a set S to have the property that all continuous functions on it be bounded is that S be compact. We've just shown that (7) is also necessary and sufficient, so this establishes the following theorems.

THEOREM 6.5. A set $S \subseteq R$ is compact if and only if every point of S^* is infinitely close to a point of S.

THEOREM 6.6 If the standard function $f(x)$ is continuous on a standard compact set S, then $f(x)$ is bounded there.

It turns out that in applying the methods of Non-standard Analysis to the subject of General Topology, the characterization of compactness given by Theorem 6.5 still holds.

The theorem below gives a very nice characterization of the notion of a uniformly continuous function. We shall not deny you the pleasure of trying to prove it yourself. The proof of Theorem 6.1 should provide the inspiration.

THEOREM 6.7. A standard function is uniformly continuous on the standard set S if and only if $x \approx y$ implies $f^*(x) \approx f^*(y)$ for all $x, y \in S^*$.

Using the above theorem we can quickly dispatch the following.

THEOREM 6.8. A standard function f continuous on a compact standard set S is uniformly continuous on S.

PROOF. Let $x, y \in S^*$ be given such that $x \approx y$. By compactness of S there exists $x_0 \in S$ such that $x \approx x_0$. Since $\approx$ is an equivalence relation $x \approx x_0 \approx y$. Now by continuity $f^*(x) \approx f(x_0) \approx f^*(y)$, therefore $f^*(x) \approx f^*(y)$.

7. INFINITE SEQUENCES

An infinite sequence $\{a_n\}$ can be thought of as a function from N into R. Accordingly the Main Theorem provides for an extension function from N^* into R^*. Put differently, after we exhaust all the terms with finite subscripts, the sequence continues on with infinite subscripts as follows:

$$\underbrace{a_1, a_2, \ldots, a_n \ldots}_{\substack{\text{terms with}\\ \text{finite}\\ \text{subscripts}}} \qquad \underbrace{\ldots a_{\alpha-1}, a_\alpha, a_{\alpha+1} \ldots}_{\substack{\text{terms with}\\ \text{infinite}\\ \text{subscripts}}}$$

It is easy to see that the sequence

$$0, 0, \ldots, 0 \ldots$$

continues to have the value 0 when we look at its extension because the statement

$$(\forall x)\,(x \in N \rightarrow a_x = 0)$$

is true in R and therefore in R^*. Likewise the sequence

$$1, 0, 1, 0, \ldots, 1, 0, \ldots$$

continues to alternate, and the sequence of primes $p_1, p_2, p_3, \ldots, p_n, \ldots$ when extended "enumerates" the primes of N^*.

Various properties of standard sequences can be characterized in terms of what happens to the terms with infinite subscripts (intuitively—when you get out to infinity).

In what follows $\{a_n\}$, $\{b_n\}$ will be standard sequences and a, b will be standard numbers. The proof of the following theorem runs along lines which by now should be familiar to you.

THEOREM 7.1.

(i) $\{a_n\}$ is bounded iff a_α is finite for all infinite natural numbers α.

(ii) $\lim\limits_{n\to\infty} a_n = a$ iff $a_\alpha \approx a$ for all infinite natural numbers α.

(iii) $\lim\limits_{n\to\infty} a_n = \infty$ iff a_α is infinite for all infinite natural numbers α.

(iv) $\{a_n\}$ is a Cauchy sequence iff $a_\alpha \approx a_\beta$ for all infinite natural numbers α, β.

Example 7.1. Suppose

$$\lim_{n\to\infty} a_n = a \text{ and } \lim_{n\to\infty} b_n = b,$$

and we want to show

$$\lim_{n\to\infty} (a_n + b_n) = a + b \text{ and } \lim_{n\to\infty} a_n b_n = a\,b.$$

Let α be an infinite natural number. By the above theorem we have $a_\alpha \approx a$ and $b_\alpha \approx b$. From this we see easily that a_α and b_α are finite. Now using the rules given in Section 2 for manipulating the $\approx$ symbol,

$$a_\alpha + b_\alpha \approx a + b \text{ and } a_\alpha b_\alpha \approx a\, b.$$

Thus by the above theorem, the desired results are established.

Example 7.2. Suppose we wanted to calculate

$$\lim_{n \to \infty} (n^2 - n) = ?$$

We can proceed directly— let α be an arbitrary infinite natural number, then

$$\begin{aligned} \alpha^2 - \alpha = \alpha(\alpha - 1) &= \text{(infinite) (infinite)} \\ &= \text{infinite} \end{aligned}$$

thus

$$\lim_{n \to \infty} (n^2 - n) = \infty.$$

8. Infinitely Fine Partitions of an Interval

Consider the familiar process of partitioning an interval $[a, b]$ into n subintervals of equal length by means of the partition points

$$a = a_0 < a_1 < \cdots < a_n = b.$$

If we let a_i^j denote the i^{th} partition point when the interval is divided into j subintervals of equal length, it is easily seen that

$$a_i^j = a + \left(\frac{b-a}{j}\right) i.$$

Now the right side of this expression is a function from $I \times I$ into R, where $I \subseteq R$ is the set of integers. By the Main Theorem this function extends to a function from $I^* \times I^*$ into R^*. We continue to use a_i^j for the image under this extended function. If we let α be a fixed infinite natural number, then for $0 \leqq i \leqq \alpha$, a_i^α must lie in the interval $[a, b]^*$. Note that the i^{th} sub-interval $[a_i^\alpha, a_{i+1}^\alpha]$ has the infinitesimal $\dfrac{b-a}{\alpha}$ as its length. Two such intervals can intersect only if they have an end point in common, and the intersection is that end point. Each partition point a_i^α (other than a, b) has an immediately