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For any finite set  $T \subseteq R$  we can show  $T = T^*$ . Suppose  $T = \{a_1, ..., a_n\}$  then

$$(\forall x) \ (x \in T \leftrightarrow [x = a_1 \lor x = a_2 \lor \dots \lor x = a_n])$$

is true in R, thus

$$(\forall x) (x \in T^* \leftrightarrow [x = a_1 \lor x = a_2 \lor \dots \lor x = a_n])$$

is true in  $R^*$ ; that is,  $T^* = \{a_1, ..., a_n\}$ .

Although the Main Theorem makes no mention of functions f whose domain is a proper subset  $D \subset R$ . We can define a function  $f^* : D^* \to R^*$  in a natural way. Arbitrarily extend f to a function g which is defined on all of R; then let  $f^*$  be the restriction of  $g^*$  to  $D^*$ . This definition is easily seen to be independent of the way f is extended.

## 5. INFINITE NATURAL NUMBERS

We have seen in the last section that each particular  $S \subseteq R$  has associated with it a certain extension  $S^* \subseteq R^*$ . We now consider the case when we take S to be N, the set of natural numbers. One can see that  $N^*$  actually has some non-standard members as follows. The statement "N is unbounded" is true in R and can be formulated as the admissible statement

$$(\forall x) (\exists y) (y \in N \land y > x);$$

therefore

$$(\forall x) (\exists y) (y \in N^* \land y > x)$$

is true in  $R^*$ . It asserts that  $N^*$  is an unbounded subset of  $R^*$ . If we let  $\alpha$  be an infinite member of  $R^*$ , then  $N^*$  must have an even larger member which, of course, is also infinite and non-standard.

We can show that all the non-standard members of N \* are infinite in the following way. Formulate as admissible statements each of the infinitely many assertions:

"All natural numbers are greater than 0." "No natural numbers lie between 0 and 1." "No natural numbers lie between 1 and 2." etc. Each of those statements then must be true in  $R^*$  when we read  $N^*$  instead of N, so each member of  $N^* - N$  must be greater than all the real numbers.

In view of the above we call the non-standard members of N \* infinite natural numbers.

Now it is easy to show that each infinite natural number has an immediate successor in  $N^*$  (because of the corresponding result for N), and each infinite natural number has an infinite immediate predecessor in  $N^*$ .  $N^*$  isn't well ordered because if  $\alpha$  is an infinite natural number, the chain

 $\alpha > \alpha - 1 > \alpha - 2 > \cdots$ 

has no least member. Here again one might be tempted to use the Main Theorem to infer that N \* is well ordered because N is; however, the statement that N is well ordered is not admissible by virtue of its having a variable ranging over subsets. It reads:

'Every non-empty subset of 
$$N \dots$$
''

Concepts such as even number, odd number, and prime number are all meaningful for infinite natural numbers; indeed, if  $E \subseteq N$  is the set of even numbers, then  $E^*$  is the set of even numbers of  $N^*$ .

It will be shown later that  $N^*$  is uncountably infinite.

# 6. LIMITS, CONTINUITY, BOUNDEDNESS, AND COMPACTNESS

Now we show that  $R^*$  provides the appropriate machinery for formulating concepts from the Calculus in an intuitive and direct way. Consider, for example, the limit concept. The  $\varepsilon - \delta$  definition of  $\lim_{x \to c} f(x) = L$  seems to be a roundabout way of saying that for x infinitely close to but not equal to c, f(x) will be infinitely close to L. Now it makes sense to say it just that way provided we are talking about  $f^*(x)$ . It not only makes sense, but as the next theorem shows, saying it that way actually gives a correct characterization of  $\lim_{x \to c} f(x) = L$ .

THEOREM 6.1. Let f be a standard function defined on a standard open interval (a, b) having c as an interior point. Suppose further that L is standard, then

(a)  $\lim_{x \to c} f(x) = L$  if and only if  $c \neq x \approx c$  implies  $f^*(x) \approx L$ .