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For any finite set $T \subseteq R$ we can show $T = T^*$. Suppose $T = \{a_1, ..., a_n\}$ then

$$(\forall x)\,(x \in T \leftrightarrow [x = a_1 \vee x = a_2 \vee \dots \vee x = a_n])$$

is true in R, thus

$$(\forall x)\,(x \in T^* \leftrightarrow [x = a_1 \vee x = a_2 \vee \dots \vee x = a_n])$$

is true in R^*; that is, $T^* = \{a_1, ..., a_n\}$.

Although the Main Theorem makes no mention of functions f whose domain is a proper subset $D \subset R$. We can define a function $f^* : D^* \to R^*$ in a natural way. Arbitrarily extend f to a function g which is defined on all of R; then let f^* be the restriction of g^* to D^*. This definition is easily seen to be independent of the way f is extended.

5. Infinite Natural Numbers

We have seen in the last section that each particular $S \subseteq R$ has associated with it a certain extension $S^* \subseteq R^*$. We now consider the case when we take S to be N, the set of natural numbers. One can see that N^* actually has some non-standard members as follows. The statement "N is unbounded" is true in R and can be formulated as the admissible statement

$$(\forall x)\,(\exists y)\,(y \in N \wedge y > x)\,;$$

therefore

$$(\forall x)\,(\exists y)\,(y \in N^* \wedge y > x)$$

is true in R^*. It asserts that N^* is an unbounded subset of R^*. If we let α be an infinite member of R^*, then N^* must have an even larger member which, of course, is also infinite and non-standard.

We can show that all the non-standard members of N^* are infinite in the following way. Formulate as admissible statements each of the infinitely many assertions:

"All natural numbers are greater than 0."
"No natural numbers lie between 0 and 1."
"No natural numbers lie between 1 and 2."
etc.

Each of those statements then must be true in R^* when we read N^* instead of N, so each member of $N^* - N$ must be greater than all the real numbers.

In view of the above we call the non-standard members of N^* *infinite natural numbers.*

Now it is easy to show that each infinite natural number has an immediate successor in N^* (because of the corresponding result for N), and each infinite natural number has an infinite immediate predecessor in N^*. N^* isn't well ordered because if α is an infinite natural number, the chain

$$\alpha > \alpha - 1 > \alpha - 2 > \cdots$$

has no least member. Here again one might be tempted to use the Main Theorem to infer that N^* is well ordered because N is; however, the statement that N is well ordered is not admissible by virtue of its having a variable ranging over subsets. It reads:

"Every non-empty subset of N..."

Concepts such as even number, odd number, and prime number are all meaningful for infinite natural numbers; indeed, if $E \subseteq N$ is the set of even numbers, then E^* is the set of even numbers of N^*.

It will be shown later that N^* is uncountably infinite.

6. Limits, Continuity, Boundedness, and Compactness

Now we show that R^* provides the appropriate machinery for formulating concepts from the Calculus in an intuitive and direct way. Consider, for example, the limit concept. The $\varepsilon - \delta$ definition of $\lim_{x \to c} f(x) = L$ seems to be a roundabout way of saying that for x infinitely close to but not equal to c, $f(x)$ will be infinitely close to L. Now it makes sense to say it just that way provided we are talking about $f^*(x)$. It not only makes sense, but as the next theorem shows, saying it that way actually gives a correct characterization of $\lim_{x \to c} f(x) = L$.

Theorem 6.1. Let f be a standard function defined on a standard open interval (a, b) having c as an interior point. Suppose further that L is standard, then

(a) $\lim_{x \to c} f(x) = L$ if and only if $c \neq x \approx c$ implies $f^*(x) \approx L$.