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$<^*, f^*$  and the variables are to range over  $R^*$ . Sometimes we shall put on some of the stars for emphasis.

#### 4. FIXED SUBSETS

Let  $S$  be a particular (fixed) subset of  $R$ . We can identify  $S$  with the one-place relation  $S(x)$  which holds for a given  $x$  if and only if  $x \in S$ ; that is,

$$S = \{ x \in R \mid S(x) \}.$$

We can now define a set  $S^* \subseteq R^*$  by

$$S^* = \{ x \in R^* \mid S^*(x) \}.$$

Clearly  $S \subseteq S^*$  because  $S^*(x)$  agrees with  $S(x)$  on  $R$ . We shall often write

$$x \in S \text{ instead of } S(x)$$

and

$$x \in S^* \text{ instead of } S^*(x).$$

The upshot of the above is that the Main Theorem also provides for an extension  $S^*$  for each  $S \subseteq R$  and that we can allow as admissible statements those which involve the sentence fragment  $x \in S$ ; in “lifting” statements from  $R$  to  $R^*$  we replace the fragment  $x \in S$  by  $x \in S^*$ . Warning! The requirement that admissible statements be permitted only variables ranging over  $R$  hasn’t been altered. In a given statement the functions, relations, and subsets must remain fixed!

Example 4.1. Let  $S = \{ x \in R \mid x < 6 \}$ . Now

$$(\forall x) (x \in S \leftrightarrow x < 6) \text{ is true in } R$$

so

$$(\forall x) (x \in S^* \leftrightarrow x <^* 6) \text{ is true in } R^*.$$

Thus

$$S^* = \{ x \in R^* \mid x <^* 6 \}.$$

Furthermore  $S^*$  is a proper extension of  $S$ , because for any infinitesimal  $\varepsilon$ , the number  $5 + \varepsilon$  is a member of  $S^*$ , but not being a standard number, it can’t be a member of  $S$ .

For any finite set  $T \subseteq R$  we can show  $T = T^*$ . Suppose  $T = \{a_1, \dots, a_n\}$  then

$$(\forall x) (x \in T \leftrightarrow [x = a_1 \vee x = a_2 \vee \dots \vee x = a_n])$$

is true in  $R$ , thus

$$(\forall x) (x \in T^* \leftrightarrow [x = a_1 \vee x = a_2 \vee \dots \vee x = a_n])$$

is true in  $R^*$ ; that is,  $T^* = \{a_1, \dots, a_n\}$ .

Although the Main Theorem makes no mention of functions  $f$  whose domain is a proper subset  $D \subset R$ . We can define a function  $f^* : D^* \rightarrow R^*$  in a natural way. Arbitrarily extend  $f$  to a function  $g$  which is defined on all of  $R$ ; then let  $f^*$  be the restriction of  $g^*$  to  $D^*$ . This definition is easily seen to be independent of the way  $f$  is extended.

## 5. INFINITE NATURAL NUMBERS

We have seen in the last section that each particular  $S \subseteq R$  has associated with it a certain extension  $S^* \subseteq R^*$ . We now consider the case when we take  $S$  to be  $N$ , the set of natural numbers. One can see that  $N^*$  actually has some non-standard members as follows. The statement “ $N$  is unbounded” is true in  $R$  and can be formulated as the admissible statement

$$(\forall x) (\exists y) (y \in N \wedge y > x);$$

therefore

$$(\forall x) (\exists y) (y \in N^* \wedge y > x)$$

is true in  $R^*$ . It asserts that  $N^*$  is an unbounded subset of  $R^*$ . If we let  $\alpha$  be an infinite member of  $R^*$ , then  $N^*$  must have an even larger member which, of course, is also infinite and non-standard.

We can show that all the non-standard members of  $N^*$  are infinite in the following way. Formulate as admissible statements each of the infinitely many assertions:

“All natural numbers are greater than 0.”

“No natural numbers lie between 0 and 1.”

“No natural numbers lie between 1 and 2.”

etc.