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## 3. The Main Theorem

All that has been said here so far was known long before the advent of Non-Standard Analysis. Now we come to the heart of the matter—the key theorem. It was first obtained by Robinson as a corollary to the so-called Compactness Theorem of mathematical logic. Later proofs were given by means of the ultraproduct construction which also has its roots in mathematical logic. We shall content ourselves with a mere statement of the result. $R$ as usual denotes the real number system and $N$ the natural number system.

Theorem 3.1 (Main theorem). There is a set $R^*$ for which all of the following hold:

1. $R$ is a proper subset of $R^*$.
2. To each $n$-place function $f(x_1, ..., x_n)$ from $R^n$ to $R$ ($n \geqq 1$), there corresponds a certain function $f^*(x_1, ..., x_n)$ from $(R^*)^n$ to $R^*$ which agrees with $f(x_1, ..., x_n)$ on $R^n$.
3. To each $n$-place relation $A(x_1, ..., x_n)$ on $R$ ($n \geqq 1$), there corresponds a certain relation $A^*(x_1, ..., x_n)$ on $R^*$ which agrees with $A(x_1, ..., x_n)$ on $R$. The relation corresponding to the equality relation on $R$ is the equality relation on $R^*$.
4. Every statement $\mathscr{S}$ formulated in terms of
   i) particular (fixed) real numbers
   ii) particular (fixed) real functions
   iii) particular (fixed) real relations
   iv) variables ranging over $R$
   v) logical operations and quantifiers

   is true about $R$ if and only if the statement $\mathscr{S}^*$ obtained from it by
   a) replacing each $f(x_1, ..., x_n)$ by $f^*(x_1, ..., x_n)$
   b) replacing each $A(x_1, ..., x_n)$ by $A^*(x_1, ..., x_n)$
   c) letting the variables range over $R^*$

   is true about $R^*$.

It turns out that there are many such $R^*$. From here on out it will be assumed that we are fixing on one of them.

The theorem is quite a mouthful and it must be admitted that our formulation of it suffers from a little imprecision owing to the fact that we never said what a statement is [1]). A few examples, however, should nail the idea down. Let us add for emphasis that we are only allowing statements of finite length.

Example 3.1. Consider the statement

$$(\forall x)\,(0 + x = x)$$

which is true when the variables range over $R$; it asserts that the particular real number 0 is a left identity for the $+$ operation. By the Main Theorem, the statement

$$(\forall x)\,(0 +^* x = x)$$

must be true when the variables range over $R^*$; thus 0 is also a left identity for the $+^*$ operation on $R^*$.

Example 3.2. Let $f$ be a particular function from $R$ to $R$ which is an "onto" function. Then the statement

$$(\forall y)\,(\exists x)\,(f(x) = y)$$

is true when the variables range over $R$. Therefore by the Main Theorem the statement

$$(\forall y)\,(\exists x)\,(f^*(x) = y)$$

is true when the variables range over $R^*$; that is, the function $f^*$ is onto $R^*$.

Henceforth instead of saying "true when the variables range over $R$", we shall simply say "true in $R$".

In subsequent discussions members of $R$ will be called *standard* numbers, while members of $R^* - R$ will be called *non-standard* numbers. Likewise functions from $R^n$ to $R$ ($n \geqq 1$), relations on $R$, and subsets of $R$ will be called *standard* functions, relations and subsets. Some writers refer to members of $R^*$ as real numbers, but we shall reserve the term for members of $R$. Thus standard number and real number have the same meaning here.

Statements which can be formulated in the manner prescribed in the hypothesis of the Main Theorem are called *admissible* statements. You should convince yourself, by writing them out if necessary, that all the axioms of an ordered field are admissible; moreover, they are true about $R$ (because

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[1]) Using the terminology of formal logic the class of statements in question can be defined as the class of closed well-formed formulae of a generalized first-order language having distinct individual, function, and relation constants corresponding to each real, real function and real relation.

$R$ is an ordered field). Now by the Main Theorem they are all true about $R^*$ if we put the stars on the symbols $+$, $\times$, $<$. But this is just a way of saying that $R^*$ is an ordered field with respect to $+^*$, $\times^*$, $<^*$. Moreover since the theorem provided that these agree with $+$, $\times$, $<$ respectively on $R$, we can say that $R^*$ is an ordered field which has $R$ as a proper subordered field. Now recalling results from our review on ordered fields we have that $R^*$ is non-Archimedean and is not complete.

Now at this point you might be getting a bit suspicious. You might ask: "Why not show the completeness of $R^*$ (and thus get a paradox) by taking the assertion that $R$ is complete, and then use the Main Theorem to conclude that $R^*$ is complete?" The catch is that the Completeness Axiom has a logical structure fundamentally different from the ordered field axioms. It's not an admissible statement! Its form is

$$(\forall S)\,(S \text{ bounded } \rightarrow \cdots\cdots\cdots)$$

that is, it has a variable ranging over the family of *subsets* of $R$. Recall, the variables in an admissible statement must range over $R$.

With respect to the Archimedean property the catch is a little different. Using the symbols $N(y)$ to denote the particular one-place relation—"$y$ is a natural number," we *can* assert that $R$ is Archimedean by the admissible statement

$$(\forall x)\,(\exists y)\,(N(y) \wedge x < y);$$

thus

$$(\forall x)\,(\exists y)\,(N^*(y) \wedge x <^* y)$$

is true in $R^*$, but it doesn't necessarily say that $R^*$ is Archimedean. The $y$ which is asserted to exist, and for which $N^*(y)$ holds, might be in $R^* - R$; that is, it might be non-standard. To be sure, it does say that $R^*$ has some sort of formal Archimedean-like property, but if in the definition of Archimedean one requires that $y$ actually be a member of $N$ (and we shall), then $R^*$ isn't Archimedean.

In the sequel it may at times be too repetitious to write statements first without the stars *, and then with them. It will usually be clear from the context whether the stars are intended. Thus if we were to say that

$$(\forall x)\,(\forall y)\,(x < y \rightarrow f(x) < f(y))$$

is true in $R^*$, then you are to understand that we are really talking about

$<^*, f^*$ and the variables are to range over $R^*$. Sometimes we shall put on some of the stars for emphasis.

## 4. Fixed Subsets

Let $S$ be a particular (fixed) subset of $R$. We can identify $S$ with the one-place relation $S(x)$ which holds for a given $x$ if and only if $x \in S$; that is,

$$S = \{ x \in R \mid S(x) \}.$$

We can now define a set $S^* \subseteq R^*$ by

$$S^* = \{ x \in R^* \mid S^*(x) \}.$$

Clearly $S \subseteq S^*$ because $S^*(x)$ agrees with $S(x)$ on $R$. We shall often write

$$x \in S \text{ instead of } S(x)$$

and

$$x \in S^* \text{ instead of } S^*(x).$$

The upshot of the above is that the Main Theorem also provides for an extension $S^*$ for each $S \subseteq R$ and that we can allow as admissible statements those which involve the sentence fragment $x \in S$; in "lifting" statements from $R$ to $R^*$ we replace the fragment $x \in S$ by $x \in S^*$. Warning! The requirement that admissible statements be permitted only variables ranging over $R$ hasn't been altered. In a given statement the functions, relations, and subsets must remain fixed!

Example 4.1. Let $S = \{ x \in R \mid x < 6 \}$. Now

$$(\forall x)\,(x \in S \leftrightarrow x < 6) \text{ is true in } R$$

so

$$(\forall x)\,(x \in S^* \leftrightarrow x <^* 6) \text{ is true in } R^*.$$

Thus

$$S^* = \{ x \in R^* \mid x <^* 6 \}.$$

Furthermore $S^*$ is a proper extension of $S$, because for any infinitesimal $\varepsilon$, the number $5 + \varepsilon$ is a member of $S^*$, but not being a standard number, it can't be a member of $S$.