

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 20 (1974)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: NON-STANDARD ANALYSIS: AN EXPOSITION
Autor: Levitz, Hilbert
Kapitel: 1. Ordered Fields
DOI: <https://doi.org/10.5169/seals-46892>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 08.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

1. ORDERED FIELDS

In this section we shall review some well known results about ordered fields and state (and prove) some not so well known ones.

1. Every ordered field contains the rational number system Q as a subordered field.
2. The real number system R constitutes a complete (in the upper bound sense) ordered field.
3. Any two complete ordered fields are isomorphic with respect to $+$, \times , and $<$.
4. Any complete ordered field R' is Archimedean; that is, to each $a \in R'$ there exists a natural number n such that $n > a$.
5. There exists ordered fields which contain the real number system as a proper subordered field.
6. Any ordered field F which contains the reals as a proper subordered field must be non-Archimedean and, consequently, cannot be complete.

PROOFS:

The first four are to be found in most advanced calculus books.

In 5 the existence of the desired field can be shown by considering $R(X)$, the field of rational functions in one indeterminate with real coefficients. R can be identified with the polynomials of degree 0. To define an ordering on $R(X)$ it is sufficient to specify the positive members, then we can define the ordering $<$ by the rule: $\alpha < \beta$ iff $\alpha - \beta$ is positive. Take for the positive elements those rational functions which can be represented as a quotient of two polynomials both of which have positive leading coefficient. This particular ordered field will play no role, however, in our subsequent discussions.

We prove 6 by contradiction. Suppose F is Archimedean. Choose α such that $\alpha \in F$ and $\alpha \notin R$. Since F is Archimedean there exists a natural number n such that $|\alpha| < n$. (Recall that the notion of absolute value is meaningful in any ordered field.) Let $A = \{x \in R \mid x \leq |\alpha|\}$. A is bounded above by n , so A has a smallest real upper bound s . Now since s is real and α isn't, we have that $s \neq |\alpha|$ and we can form the reciprocal of $s - |\alpha|$. Since F is assumed to be Archimedean, there exists a natural number k such that

$$k > \frac{1}{|s - |\alpha||}$$

and from this we get

$$s - |\alpha| > \frac{1}{k} \quad \text{or} \quad |\alpha| - s > \frac{1}{k}.$$

Case 1. $s - |\alpha| > \frac{1}{k}$. In this case $s - \frac{1}{k} > |\alpha|$, but then by definition of A we see that $s - \frac{1}{k}$ is a real upper bound of A . Moreover, $s - \frac{1}{k}$ is smaller than the *least* upper bound s , which is absurd.

Case 2. $|\alpha| - s > \frac{1}{k}$. Then $|\alpha| > s + \frac{1}{k}$, so $s + \frac{1}{k} \in A$ by definition of A . But s is an upper bound for A so $s + \frac{1}{k} \leq s$ from which follows $k \leq 0$; but this contradicts the fact that k is a natural number.

(Q.E.D.)

2. ORDERED FIELDS WHICH PROPERLY CONTAIN THE REALS

In this section we shall assume that F is an ordered field which has the real numbers R as a proper subordered field. We have already seen that F must be non-Archimedean. N will be used to denote the set of natural numbers.

An element $a \in F$ is said to be

- infinitesimal* if $|a| < r$ for each positive real r .
- finite* if $|a| \leq r$ for some real r .
- infinite* if $|a| > r$ for every real r .

The number 0 is certainly infinitesimal, but it is easy to see that there are also non-zero infinitesimals and infinites as follows:

F being non-Archimedean must contain an element b such that $n \leq b$ for all $n \in N$. This implies that $n < b$ all $n \in N$ and, in fact, $r < b$ all $r \in R$.

Thus b is infinite and $\frac{1}{b}$ is infinitesimal.