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NON-STANDARD ANALYSIS: AN EXPOSITION

by Hilbert LEVITZ

In 1960 Abraham Robinson solved the age old problem of producing a calculus involving infinitely small (infinitesimal) and infinitely large quantities. As is well known, Leibnitz and the early developers of the Calculus freely used infinitesimals, but this usage was abandoned during the last century because of its apparent lack of a logical foundation. The need for infinitesimals was completely circumvented by the familiar $\varepsilon - \delta$ methods of Cauchy and Weirstrass. In this way logical rigor was obtained but at the expense of having to tolerate a certain indirectness of expression. Now Robinson has shown that we can have rigor and infinitesimals too. Within the framework in which he has set up the Calculus, it is meaningful to refer to the values that a function $f(x)$ assumes for x values infinitely close to, yet not equal, a given point x_0 in its domain.

During the past decade under the leadership of Robinson and W. A. J. Luxemburg a small but growing number of researchers have been applying these methods to other mathematical disciplines among them algebra, topology, number theory, and probability. The mathematical community at large has only recently begun to take note of this work, and the number of persons who can employ infinitesimals with confidence is still not very great. This may have something to do with the fact that this development has taken place within the context of mathematical logic, a subject whose methods many mathematicians find uncongenial and whose most fundamental results have not yet become common knowledge for mathematics graduates.

In this article we will play down the precise terminology of mathematical logic in favor of a kind of informal usage which should be adequate for conveying the ideas involved. To be sure, we won't prove the key existence theorem on which Robinson's "Non-Standard Analysis" is based. Our aim is simply to give you a good operational feel for what is legitimate in doing analysis with infinitesimals and infinites. Anyone who has survived $\varepsilon - \delta$ proofs should be able to pick up some technical facility here for correctly apply these concepts to problem solving and theorem proving in the Calculus.

1. Ordered Fields

In this section we shall review some well known results about ordered fields and state (and prove) some not so well known ones.

1. Every ordered field contains the rational number system Q as a subordered field.
2. The real number system R constitutes a complete (in the upper bound sense) ordered field.
3. Any two complete ordered fields are isomorphic with respect to $+$, $\times$, and $<$.
4. Any complete ordered field R' is Archimedean; that is, to each $a \in R'$ there exists a natural number n such that $n > a$.
5. There exists ordered fields which contain the real number system as a proper subordered field.
6. Any ordered field F which contains the reals as a proper subordered field must be non-Archimedean and, consequently, cannot be complete.

Proofs:

The first four are to be found in most advanced calculus books.

In 5 the existence of the desired field can be shown by considering $R(X)$, the field of rational functions in one indeterminate with real coefficients. R can be identified with the polynomials of degree 0. To define an ordering on $R(X)$ it is sufficient to specify the positive members, then we can define the ordering $<$ by the rule: $\alpha < \beta$ iff $\alpha - \beta$ is positive. Take for the positive elements those rational functions which can be represented as a quotient of two polynomials both of which have positive leading coefficient. This particular ordered field will play no role, however, in our subsequent discussions.

We prove 6 by contradiction. Suppose F is Archimedean. Choose α such that $\alpha \in F$ and $\alpha \notin R$. Since F is Archimedean there exists a natural number n such that $|\alpha| < n$. (Recall that the notion of absolute value is meaningful in any ordered field.) Let $A = \{x \in R \mid x \leqq |\alpha|\}$. A is bounded above by n, so A has a smallest real upper bound s. Now since s is real and α isn't, we have that $s \neq |\alpha|$ and we can form the reciprocal of $s - |\alpha|$. Since F is assumed to be Archimedean, there exists a natural number k such that

$$k > \frac{1}{|s - |\alpha||}$$

and from this we get

$$s - |\alpha| > \frac{1}{k} \quad or \quad |\alpha| - s > \frac{1}{k}.$$

Case 1. $s - |\alpha| > \frac{1}{k}$. In this case $s - \frac{1}{k} > |\alpha|$, but then by definition of A we see that $s - \frac{1}{k}$ is a real upper bound of A. Moreover, $s - \frac{1}{k}$ is smaller than the *least* upper bound s, which is absurd.

Case 2. $|\alpha| - s > \frac{1}{k}$. Then $|\alpha| > s + \frac{1}{k}$, so $s + \frac{1}{k} \in A$ by definition of A. But s is an upper bound for A so $s + \frac{1}{k} \leqq s$ from which follows $k \leqq 0$; but this contradicts the fact that k is a natural number.

(Q.E.D.)

2. Ordered Fields Which Properly Contain the Reals

In this section we shall assume that F is an ordered field which has the real numbers R as a proper subordered field. We have already seen that F must be non-Archimedean. N will be used to denote the set of natural numbers.

An element $a \in F$ is said to be

infinitesimal if $|a| < r$ for each positive real r.

finite if $|a| \leqq r$ for some real r.

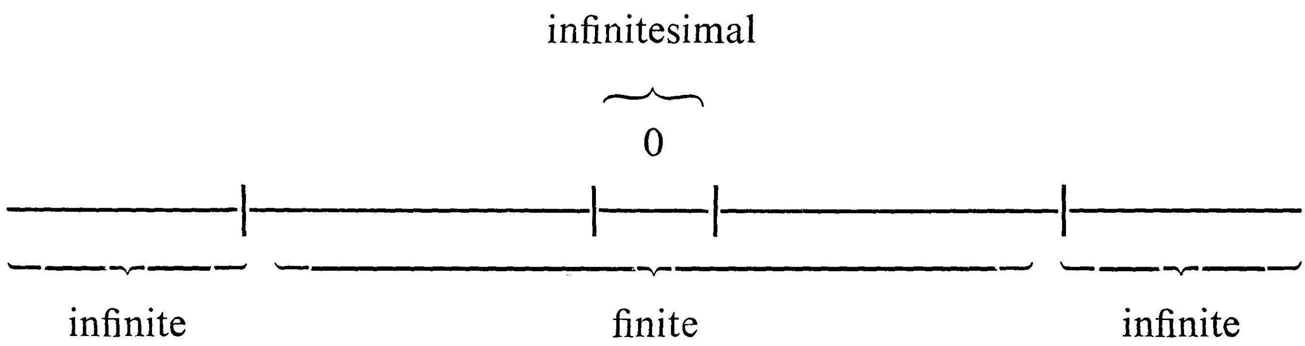
infinite if $|a| > r$ for every real r.

The number 0 is certainly infinitesimal, but it is easy to see that there are also non-zero infinitesimals and infinites as follows:

F being non-Archimedean must contain an element b such that $n \leqq b$ for all $n \in N$. This implies that $n < b$ all $n \in N$ and, in fact, $r < b$ all $r \in R$. Thus b is infinite and $\frac{1}{b}$ is infinitesimal.

Whenever α and β differ by an infinitesimal we say that α is *near* β or α is *infinitely close* to β. We symbolize this $\alpha \approx \beta$. The relation $\approx$ is easily seen to be an equivalence relation. The infinitesimals are precisely those numbers which are infinitely close to 0.

A crude picture of F appears below.



Note that infinitesimals are finite, and reals are finite; in fact, all numbers of the form real + infinitesimal are finite. It turns out that the finite numbers can always be written this way and we formulate this as a theorem below.

THEOREM 2.1. Every finite number can be written in the form

real + infinitesimal.

Moreover, this representation is unique. Put differently, every finite number is infinitely close to a uniquely determined real number.

PROOF:

Uniqueness of representation—suppose that

$$r_1 + \varepsilon_1 = r_2 + \varepsilon_2$$

where $r_1, r_2 \in R$ and $\varepsilon_1, \varepsilon_2$ are infinitesimal. Then

$$r_1 - r_2 = \varepsilon_2 - \varepsilon_1.$$

Now the left side is real and it is easy to see that the right side is infinitesimal. But the only real which is infinitesimal is zero, so both sides are zero; thus

$$r_1 = r_2 \text{ and } \varepsilon_1 = \varepsilon_2.$$

Existence of representation—let α be finite, to show that $\alpha = a + \varepsilon$ where a is real and ε is infinitesimal. Let $A = \{ x \in R \mid x < \alpha \}$. Since α is finite there exists $r \in R$ such that $\alpha < r$. Now A is bounded by r, so it has a real least upper bound a.

Case 1. $|\alpha - a| = s$ for some $s \in R$. Then α has the desired representation $\alpha = (a \pm s) + 0$.

Case 2. $|\alpha - a|$ is not real. Now assert (and will show) that $\alpha - a$ is infinitesimal; then α would have the desired representation $\alpha = a + (\alpha - a)$. We show that $\alpha - a$ is infinitesimal by contradiction. Suppose not, then $|\alpha - a| \geqq s > 0$ for some $s \in R$.

Case 2.1. $a - \alpha > s$. Then $a - s > \alpha$ so $a - s$ is a real upper bound for A. But a is the least upper bound, so $a \leqq a - s$. From this follows $0 \geqq s$ which contradicts s being positive.

Case 2.2. $\alpha - a > s$. Then $\alpha > a + s$, so $a + s \in A$. But a is an upper bound for A, so $a \geqq a + s$. Thus $0 \geqq s$ which contradicts s being positive.

(Q.E.D.)

The following rules are all easy to verify:

$$\text{finite} \pm \text{finite} = \text{finite}$$

$$\text{infinitesimal} \pm \text{infinitesimal} = \text{infinitesimal}$$

$$\text{finite} \pm \text{infinite} = \text{infinite}$$

$$\text{finite} \cdot \text{finite} = \text{finite}$$

$$\text{infinitesimal} \cdot \text{finite} = \text{infinitesimal}$$

$$\text{infinite} \cdot \text{infinite} = \text{infinite}$$

$$\frac{1}{\text{infinite}} = \text{infinitesimal}$$

$$\frac{1}{\text{non-zero infinitesimal}} = \text{infinite}$$

These rules tell us, among other things, that the finite numbers constitute a ring and the infinitesimals constitute an ideal in this ring.

The situations below are indeterminate:

$$\text{infinite} \pm \text{infinite} = ?$$

$$\text{finite} \cdot \text{infinite} = ?$$

$$\frac{\text{infinitesimal}}{\text{infinitesimal}} = ?$$

$$\frac{\text{infinite}}{\text{infinite}} = ?$$

$$\frac{\text{finite}}{\text{finite}} = ?$$

Note that since we are working in a field, $0 \cdot \text{infinite}$ is actually defined and has the value 0.

The infinitely close symbol $\approx$ can be manipulated as follows:

If $\alpha \approx \beta$ and $u \approx v$, then

$$\alpha + u \approx \beta + v$$

$$\alpha - u \approx \beta - v$$

$$\alpha \cdot u \approx \beta \cdot v \text{ (provided } \alpha, u \text{ are finite)}$$

$$\frac{\alpha}{u} \approx \frac{\beta}{v} \quad \text{(provided } \alpha \text{ is finite and } u \text{ ins't infinitesimal)}$$

The hint to proving the last two is to write $\alpha = \beta + \varepsilon$, and $u = v + \delta$ where ε, δ are infinitesimals.

We have already seen that every finite number α is infinitely close to a uniquely determined real number; we call this real number the *standard part* of α and denote it by $^{\circ}\alpha$.

The following are obtained from the rules just given (α, β are assumed to be finite):

$$^{\circ}(\alpha+\beta) = {^{\circ}\alpha} + {^{\circ}\beta}$$

$$^{\circ}(\alpha-\beta) = {^{\circ}\alpha} - {^{\circ}\beta}$$

$$^{\circ}(\alpha \cdot \beta) = {^{\circ}\alpha} \cdot {^{\circ}\beta}$$

$$^{\circ}\left(\frac{\alpha}{\beta}\right) = \frac{^{\circ}\alpha}{^{\circ}\beta} \quad \text{(provided } \beta \text{ isn't infinitesimal).}$$

Remember now! The symbol $^{\circ}\alpha$ only makes sense when α is finite.

3. The Main Theorem

All that has been said here so far was known long before the advent of Non-Standard Analysis. Now we come to the heart of the matter—the key theorem. It was first obtained by Robinson as a corollary to the so-called Compactness Theorem of mathematical logic. Later proofs were given by means of the ultraproduct construction which also has its roots in mathematical logic. We shall content ourselves with a mere statement of the result. R as usual denotes the real number system and N the natural number system.

Theorem 3.1 (Main theorem). There is a set R^* for which all of the following hold:

1. R is a proper subset of R^*.
2. To each n-place function $f(x_1, ..., x_n)$ from R^n to R ($n \geqq 1$), there corresponds a certain function $f^*(x_1, ..., x_n)$ from $(R^*)^n$ to R^* which agrees with $f(x_1, ..., x_n)$ on R^n.
3. To each n-place relation $A(x_1, ..., x_n)$ on R ($n \geqq 1$), there corresponds a certain relation $A^*(x_1, ..., x_n)$ on R^* which agrees with $A(x_1, ..., x_n)$ on R. The relation corresponding to the equality relation on R is the equality relation on R^*.
4. Every statement $\mathcal{S}$ formulated in terms of

   i) particular (fixed) real numbers
   ii) particular (fixed) real functions
   iii) particular (fixed) real relations
   iv) variables ranging over R
   v) logical operations and quantifiers

   is true about R if and only if the statement $\mathcal{S}^*$ obtained from it by

   a) replacing each $f(x_1, ..., x_n)$ by $f^*(x_1, ..., x_n)$
   b) replacing each $A(x_1, ..., x_n)$ by $A^*(x_1, ..., x_n)$
   c) letting the variables range over R^*

   is true about R^*.

It turns out that there are many such R^*. From here on out it will be assumed that we are fixing on one of them.

The theorem is quite a mouthful and it must be admitted that our formulation of it suffers from a little imprecision owing to the fact that we never said what a statement is [1]). A few examples, however, should nail the idea down. Let us add for emphasis that we are only allowing statements of finite length.

Example 3.1. Consider the statement

$$(\forall x)\,(0 + x = x)$$

which is true when the variables range over R; it asserts that the particular real number 0 is a left identity for the $+$ operation. By the Main Theorem, the statement

$$(\forall x)\,(0 +^{*}\; x = x)$$

must be true when the variables range over R^*; thus 0 is also a left identity for the $+^*$ operation on R^*.

Example 3.2. Let f be a particular function from R to R which is an "onto" function. Then the statement

$$(\forall y)\,(\exists x)\,(f(x) = y)$$

is true when the variables range over R. Therefore by the Main Theorem the statement

$$(\forall y)\,(\exists x)\,(f^*(x) = y)$$

is true when the variables range over R^*; that is, the function f^* is onto R^*.

Henceforth instead of saying "true when the variables range over R", we shall simply say "true in R".

In subsequent discussions members of R will be called *standard* numbers, while members of $R^* - R$ will be called *non-standard* numbers. Likewise functions from R^n to R ($n \geq 1$), relations on R, and subsets of R will be called *standard* functions, relations and subsets. Some writers refer to members of R^* as real numbers, but we shall reserve the term for members of R. Thus standard number and real number have the same meaning here.

Statements which can be formulated in the manner prescribed in the hypothesis of the Main Theorem are called *admissible* statements. You should convince yourself, by writing them out if necessary, that all the axioms of an ordered field are admissible; moreover, they are true about R (because

[1]) Using the terminology of formal logic the class of statements in question can be defined as the class of closed well-formed formulae of a generalized first-order language having distinct individual, function, and relation constants corresponding to each real, real function and real relation.

R is an ordered field). Now by the Main Theorem they are all true about R^* if we put the stars on the symbols $+$, $\times$, $<$. But this is just a way of saying that R^* is an ordered field with respect to $+^*$, $\times^*$, $<^*$. Moreover since the theorem provided that these agree with $+$, $\times$, $<$ respectively on R, we can say that R^* is an ordered field which has R as a proper subordered field. Now recalling results from our review on ordered fields we have that R^* is non-Archimedean and is not complete.

Now at this point you might be getting a bit suspicious. You might ask: "Why not show the completeness of R^* (and thus get a paradox) by taking the assertion that R is complete, and then use the Main Theorem to conclude that R^* is complete?" The catch is that the Completeness Axiom has a logical structure fundamentally different from the ordered field axioms. It's not an admissible statement! Its form is

$$(\forall S)\,(S \text{ bounded} \rightarrow \cdots\cdots\cdots)$$

that is, it has a variable ranging over the family of *subsets* of R. Recall, the variables in an admissible statement must range over R.

With respect to the Archimedean property the catch is a little different. Using the symbols $N(y)$ to denote the particular one-place relation—"y is a natural number," we *can* assert that R is Archimedean by the admissible statement

$$(\forall x)\,(\exists y)\,(N(y) \wedge x < y);$$

thus

$$(\forall x)\,(\exists y)\,(N^*(y) \wedge x <^* y)$$

is true in R^*, but it doesn't necessarily say that R^* is Archimedean. The y which is asserted to exist, and for which $N^*(y)$ holds, might be in $R^* - R$; that is, it might be non-standard. To be sure, it does say that R^* has some sort of formal Archimedean-like property, but if in the definition of Archimedean one requires that y actually be a member of N (and we shall), then R^* isn't Archimedean.

In the sequel it may at times be too repetitious to write statements first without the stars *, and then with them. It will usually be clear from the context whether the stars are intended. Thus if we were to say that

$$(\forall x)\,(\forall y)\,(x < y \rightarrow f(x) < f(y))$$

is true in R^*, then you are to understand that we are really talking about

$<^*, f^*$ and the variables are to range over R^*. Sometimes we shall put on some of the stars for emphasis.

4. Fixed Subsets

Let S be a particular (fixed) subset of R. We can identify S with the one-place relation $S(x)$ which holds for a given x if and only if $x \in S$; that is,

$$S = \{ x \in R \mid S(x) \}.$$

We can now define a set $S^* \subseteq R^*$ by

$$S^* = \{ x \in R^* \mid S^*(x) \}.$$

Clearly $S \subseteq S^*$ because $S^*(x)$ agrees with $S(x)$ on R. We shall often write

$$x \in S \text{ instead of } S(x)$$

and

$$x \in S^* \text{ instead of } S^*(x).$$

The upshot of the above is that the Main Theorem also provides for an extension S^* for each $S \subseteq R$ and that we can allow as admissible statements those which involve the sentence fragment $x \in S$; in "lifting" statements from R to R^* we replace the fragment $x \in S$ by $x \in S^*$. Warning! The requirement that admissible statements be permitted only variables ranging over R hasn't been altered. In a given statement the functions, relations, and subsets must remain fixed!

Example 4.1. Let $S = \{ x \in R \mid x < 6 \}$. Now

$$(\forall x)\,(x \in S \leftrightarrow x < 6) \text{ is true in } R$$

so

$$(\forall x)\,(x \in S^* \leftrightarrow x <^* 6) \text{ is true in } R^*.$$

Thus

$$S^* = \{ x \in R^* \mid x <^* 6 \}.$$

Furthermore S^* is a proper extension of S, because for any infinitesimal ε, the number $5 + \varepsilon$ is a member of S^*, but not being a standard number, it can't be a member of S.

For any finite set $T \subseteq R$ we can show $T = T^*$. Suppose $T = \{a_1, ..., a_n\}$ then

$$(\forall x)\,(x \in T \leftrightarrow [x = a_1 \lor x = a_2 \lor \ldots \lor x = a_n])$$

is true in R, thus

$$(\forall x)\,(x \varepsilon T^* \leftrightarrow [x = a_1 \lor x = a_2 \lor \ldots \lor x = a_n])$$

is true in R^*; that is, $T^* = \{a_1, ..., a_n\}$.

Although the Main Theorem makes no mention of functions f whose domain is a proper subset $D \subset R$. We can define a function $f^* : D^* \to R^*$ in a natural way. Arbitrarily extend f to a function g which is defined on all of R; then let f^* be the restriction of g^* to D^*. This definition is easily seen to be independent of the way f is extended.

5. Infinite Natural Numbers

We have seen in the last section that each particular $S \subseteq R$ has associated with it a certain extension $S^* \subseteq R^*$. We now consider the case when we take S to be N, the set of natural numbers. One can see that N^* actually has some non-standard members as follows. The statement "N is unbounded" is true in R and can be formulated as the admissible statement

$$(\forall x)\,(\exists y)\,(y \in N \land y > x);$$

therefore

$$(\forall x)\,(\exists y)\,(y \in N^* \land y > x)$$

is true in R^*. It asserts that N^* is an unbounded subset of R^*. If we let α be an infinite member of R^*, then N^* must have an even larger member which, of course, is also infinite and non-standard.

We can show that all the non-standard members of N^* are infinite in the following way. Formulate as admissible statements each of the infinitely many assertions:

"All natural numbers are greater than 0."
"No natural numbers lie between 0 and 1."
"No natural numbers lie between 1 and 2."
etc.

Each of those statements then must be true in R^* when we read N^* instead of N, so each member of $N^* - N$ must be greater than all the real numbers.

In view of the above we call the non-standard members of N^* *infinite natural numbers.*

Now it is easy to show that each infinite natural number has an immediate successor in N^* (because of the corresponding result for N), and each infinite natural number has an infinite immediate predecessor in N^*. N^* isn't well ordered because if α is an infinite natural number, the chain

$$\alpha > \alpha - 1 > \alpha - 2 > \cdots$$

has no least member. Here again one might be tempted to use the Main Theorem to infer that N^* is well ordered because N is; however, the statement that N is well ordered is not admissible by virtue of its having a variable ranging over subsets. It reads:

"Every non-empty subset of N. . ."

Concepts such as even number, odd number, and prime number are all meaningful for infinite natural numbers; indeed, if $E \subseteq N$ is the set of even numbers, then E^* is the set of even numbers of N^*.

It will be shown later that N^* is uncountably infinite.

6. Limits, Continuity, Boundedness, and Compactness

Now we show that R^* provides the appropriate machinery for formulating concepts from the Calculus in an intuitive and direct way. Consider, for example, the limit concept. The $\varepsilon - \delta$ definition of $\lim_{x \to c} f(x) = L$ seems to be a roundabout way of saying that for x infinitely close to but not equal to c, $f(x)$ will be infinitely close to L. Now it makes sense to say it just that way provided we are talking about $f^*(x)$. It not only makes sense, but as the next theorem shows, saying it that way actually gives a correct characterization of $\lim_{x \to c} f(x) = L$.

Theorem 6.1. Let f be a standard function defined on a standard open interval (a, b) having c as an interior point. Suppose further that L is standard, then

(a) $\lim_{x \to c} f(x) = L$ if and only if $c \neq x \approx c$ implies $f^*(x) \approx L$.

(b) $f(x)$ is continuous at c if and only if $x \approx c$ implies $f^*(x) \approx f(c)$.

PROOF. At this point there should be no confusion if we sometimes omit the symbol *. Part (b) follows immediately from part (a).

Part (a), direction $\Rightarrow$. Suppose $\lim_{x \to c} f(x) = L$. To show that $c \neq x \approx c$ implies $f(x) \approx L$. Let x_0 be given such that $c \neq x_0 \approx c$, to show $f(x_0) \approx L$ we must show that $f(x_0) - L$ is infinitesimal; that is, we must show that $|f(x_0) - L| < \varepsilon$ for each positive real ε. Let arbitrary but fixed positive real ε_0 be given. Must show that the statement

(1) $$|f(x_0) - L| < \varepsilon_0$$

is true in R^*. By definition of limit we know that there exists a positive real δ such that $0 < |x - c| < \delta$ implies that $|f(x) - L| < \varepsilon_0$. Let δ_0 be such a δ, then the statement

$$(\forall x)\,(0 < |x - c| < \delta_0 \rightarrow |f(x) - L| < \varepsilon_0)$$

is true in R; therefore, it's true in R^*. In particular then the statement

(2) $$0 < |x_0 - c| < \delta_0 \rightarrow |f(x_0) - L| < \varepsilon_0$$

is true in R^*. Now from $c \neq x_0 \approx c$ we know that $0 < |x_0 - c| < r$ for each positive real r, so in particular $0 < |x_0 - c| < \delta_0$ is true in R^*. This with (2) gives $|f(x_0) - L| < \varepsilon_0$ which is the statement (1) we needed to show.

Part (a), direction $\Leftarrow$. The argument is rather novel. Assume that

(3) $$c \neq x \approx c \text{ implies } f(x) \approx L.$$

Let arbitrary but fixed positive real ε_0 be given, must show that the statement

$$(\exists \delta)\,(\delta > 0 \wedge (\forall x)\,[0 < |x - c| < \delta \rightarrow |f(x) - L| < \varepsilon_0])$$

is true in R. Now this is an admissible statement, so it suffices to show that it is true in R^*. As a statement about R it is an assertion that there exists a real δ with certain properties. In showing it to be true in R^* we are permitted to seek the δ from among the positive infinitesimals if we so desire. We now show that any positive infinitesimal δ will do. Let δ_0 be a positive infinitesimal. Must show that the statement

$$(\forall x)\,[0 < |x - c| < \delta_0 \rightarrow |f(x) - L| < \varepsilon_0]$$

is true in R^*. Let arbitrary $x_0 \in R^*$ be given, must show that

(4) $$0 < |x_0 - c| < \delta_0 \rightarrow |f(x_0) - L| < \varepsilon_0$$

is true. Assume that the left side of the arrow is true; that is, assume

(5) $$0 < |x_0 - c| < \delta_0;$$

we want to show under this assumption that the right side is true. Since δ_0 is infinitesimal we can infer from (5) that $c \neq x_0 \approx c$, but then by (3) $f(x_0) \approx L$, that is, $f(x_0) - L$ is infinitesimal. Since ε_0 is positive real, $|f(x_0) - L| < \varepsilon_0$, which is the right side of the arrow in (4)

(Q.E.D.)

Example 6.1. Suppose we want to show that the composition of two continuous functions is continuous. The standard proof is easy enough, but the following non-standard proof is more direct and intuitive. Let $g(x)$ be continuous at c and $f(x)$ continuous at $g(c)$. Let $x \approx c$ be given. Since g is continuous at c, $g(x) \approx g(c)$. Since f is continuous at $g(c)$, $f(g(x)) \approx f(g(c))$.

For functions whose domain is not an interval but some set S, appropriate modifications of the argument in the preceding theorem gives the theorem below.

THEOREM 6.2. The standard function $f(x)$ with standard domain S is continuous at the standard point c if and only if whenever x is a point of S^* infinitely close to c, $f^*(x) \approx f(c)$.

The notion of a function being bounded has a very useful non-standard characterization. By bounded we mean, as usual, that there is a standard bound.

THEOREM 6.3. A standard function f is bounded on a standard set S if and only if $f^*(x)$ is finite for each $x \in S^*$.

PROOF. Direction $\Rightarrow$. Suppose f is bounded on S. Then there exists a standard number r_0 such that the sentence

$$(\forall x)(x \in S \rightarrow |f(x)| \leq r_0)$$

is true in R and therefore also in R^*. Thus if $x_0 \in S^*$ we have that $|f^*(x_0)| \leq r_0$. By definition of finite this means $f^*(x_0)$ is finite.

Direction $\Leftarrow$. Suppose $f^*(x)$ is finite all $x \in S^*$. We want to show that the statement

(6) $$(\exists t)(\forall x)(x \in S \rightarrow |f(x)| \leq t)$$

is true in R. If suffices to show that it is true in R^*, but in R^* we can take the t to be any positive infinite number. Then since each $f^*(x)$ is finite we have that $|f^*(x)| \leqq t$ all $x \in S^*$, that is (6) is true in R^*.

(Q.E.D.)

Note that if $[c, d]$ is a standard closed interval, then $[c, d]^*$ is the closed interval $\{x \in R^* \mid c \leqq x \leqq d\}$; this is because the statement

$$(\forall x)\,(x \in [c, d] \leftrightarrow (c \leqq x \wedge x \leqq d))$$

is true in R and, therefore, in R^*. A similar result holds for the other types of intervals.

Compare the following non-standard proof of a well known theorem with the standard proofs you know!

THEOREM 6.4. If the standard function f is continuous at each point of the standard closed interval $[c, d]$, then f is bounded there.

PROOF. In view of the preceding theorem we have only to show that $f^*(x)$ is finite for all $x \in [c, d]^*$. Let $x \in [c, d]^*$ be given. Clearly x is finite and according to Theorem 2.1 it is infinitely close to a standard point x_0. It is easy to see that $x_0 \in [c, d]$. By continuity $f^*(x) \approx f(x_0)$. Since f is a standard function, $f(x_0)$ is finite, but then $f^*(x)$ being infinitely close, is also finite.

(Q.E.D.)

As is well known, the above theorem fails for open intervals. An attempted proof would break down when we try to assert that $x_0 \in (c, d)$. It might just happen that x_0 is one of the end points.

Note that in the above proof, the only property of the closed interval used there is:

"Every point of $[c, d]^*$ is infinitely closed to some point of $[c, d]$." Thus the theorem can be generalized by replacing $[c, d]$ with a set S having the same property, namely

(7) "Every point of S^* is infinitely close to some point of S."

We show further that this property of S is a necessary condition on S for *all* continuous functions on S to be bounded. By contrapositive assume (7) fails; under this assumption we will produce a continuous function on S which takes on an infinite value at a point in S^* from which it would follow that the function isn't bounded on S. To say that (7) fails would mean that there exists $x_0 \in S^*$ such that for all $y \in S$, $x_0 \not\approx y$. If x_0 is infinite, then

$f(x) = x$ is a continuous function which is infinite at $x = x_0$. On the other hand, if x_0 were finite then x_0 is infinitely close to some standard number y_0, and since (7) fails $y_0 \notin S$. Thus the function $f(x) = \dfrac{1}{x - y_0}$ is continuous on S (the denominator can't be zero for $x \in S$ because $y_0 \notin S$); moreover, $f(x_0)$ is infinite since $x_0 - y_0$ is a non-zero infinitesimal.

It is known in the study of the topology of the real line that a necessary and sufficient condition for a set S to have the property that all continuous functions on it be bounded is that S be compact. We've just shown that (7) is also necessary and sufficient, so this establishes the following theorems.

THEOREM 6.5. A set $S \subseteq R$ is compact if and only if every point of S^* is infinitely close to a point of S.

THEOREM 6.6 If the standard function $f(x)$ is continuous on a standard compact set S, then $f(x)$ is bounded there.

It turns out that in applying the methods of Non-standard Analysis to the subject of General Topology, the characterization of compactness given by Theorem 6.5 still holds.

The theorem below gives a very nice characterization of the notion of a uniformly continuous function. We shall not deny you the pleasure of trying to prove it yourself. The proof of Theorem 6.1 should provide the inspiration.

THEOREM 6.7. A standard function is uniformly continuous on the standard set S if and only if $x \approx y$ implies $f^*(x) \approx f^*(y)$ for all $x, y \in S^*$.

Using the above theorem we can quickly dispatch the following.

THEOREM 6.8. A standard function f continuous on a compact standard set S is uniformly continuous on S.

PROOF. Let $x, y \in S^*$ be given such that $x \approx y$. By compactness of S there exists $x_0 \in S$ such that $x \approx x_0$. Since $\approx$ is an equivalence relation $x \approx x_0 \approx y$. Now by continuity $f^*(x) \approx f(x_0) \approx f^*(y)$, therefore $f^*(x) \approx f^*(y)$.

7. INFINITE SEQUENCES

An infinite sequence $\{a_n\}$ can be thought of as a function from N into R. Accordingly the Main Theorem provides for an extension function from N^* into R^*. Put differently, after we exhaust all the terms with finite subscripts, the sequence continues on with infinite subscripts as follows:

$$\underbrace{a_1, a_2, \ldots, a_n \ldots}_{\substack{\text{terms with}\\ \text{finite}\\ \text{subscripts}}} \qquad \underbrace{\ldots a_{\alpha-1}, a_\alpha, a_{\alpha+1} \ldots}_{\substack{\text{terms with}\\ \text{infinite}\\ \text{subscripts}}}$$

It is easy to see that the sequence

$$0, 0, \ldots, 0 \ldots$$

continues to have the value 0 when we look at its extension because the statement

$$(\forall x)\,(x \in N \rightarrow a_x = 0)$$

is true in R and therefore in R^*. Likewise the sequence

$$1, 0, 1, 0, \ldots, 1, 0, \ldots$$

continues to alternate, and the sequence of primes $p_1, p_2, p_3, \ldots, p_n, \ldots$ when extended "enumerates" the primes of N^*.

Various properties of standard sequences can be characterized in terms of what happens to the terms with infinite subscripts (intuitively—when you get out to infinity).

In what follows $\{a_n\}$, $\{b_n\}$ will be standard sequences and a, b will be standard numbers. The proof of the following theorem runs along lines which by now should be familiar to you.

THEOREM 7.1.

(i) $\{a_n\}$ is bounded iff a_α is finite for all infinite natural numbers α.

(ii) $\lim\limits_{n\to\infty} a_n = a$ iff $a_\alpha \approx a$ for all infinite natural numbers α.

(iii) $\lim\limits_{n\to\infty} a_n = \infty$ iff a_α is infinite for all infinite natural numbers α.

(iv) $\{a_n\}$ is a Cauchy sequence iff $a_\alpha \approx a_\beta$ for all infinite natural numbers α, β.

Example 7.1. Suppose

$$\lim_{n\to\infty} a_n = a \text{ and } \lim_{n\to\infty} b_n = b,$$

and we want to show

$$\lim_{n\to\infty} (a_n + b_n) = a + b \text{ and } \lim_{n\to\infty} a_n\, b_n = a\,b.$$

Let α be an infinite natural number. By the above theorem we have $a_\alpha \approx a$ and $b_\alpha \approx b$. From this we see easily that a_α and b_α are finite. Now using the rules given in Section 2 for manipulating the $\approx$ symbol,

$$a_\alpha + b_\alpha \approx a + b \text{ and } a_\alpha\, b_\alpha \approx a\, b.$$

Thus by the above theorem, the desired results are established.

Example 7.2. Suppose we wanted to calculate

$$\lim_{n\to\infty} (n^2 - n) = ?$$

We can proceed directly— let α be an arbitrary infinite natural number, then

$$\begin{aligned} \alpha^2 - \alpha = \alpha(\alpha - 1) &= \text{(infinite) (infinite)} \\ &= \text{infinite} \end{aligned}$$

thus

$$\lim_{n\to\infty} (n^2 - n) = \infty.$$

8. Infinitely Fine Partitions of an Interval

Consider the familiar process of partitioning an interval $[a, b]$ into n subintervals of equal length by means of the partition points

$$a = a_0 < a_1 < \cdots < a_n = b.$$

If we let a_i^j denote the i^{th} partition point when the interval is divided into j subintervals of equal length, it is easily seen that

$$a_i^j = a + \left(\frac{b-a}{j}\right) i.$$

Now the right side of this expression is a function from $I \times I$ into R, where $I \subseteq R$ is the set of integers. By the Main Theorem this function extends to a function from $I^* \times I^*$ into R^*. We continue to use a_i^j for the image under this extended function. If we let α be a fixed infinite natural number, then for $0 \leqq i \leqq \alpha$, a_i^α must lie in the interval $[a, b]^*$. Note that the i^{th} subinterval $[a_i^\alpha, a_{i+1}^\alpha]$ has the infinitesimal $\dfrac{b-a}{\alpha}$ as its length. Two such intervals can intersect only if they have an end point in common, and the intersection is that end point. Each partition point a_i^α (other than a, b) has an immediately preceding partition point a_{i-1}^{α} on its left and an immediately succeeding partition point a_{i+1}^{α} on its right. One can show that each point of $[a, b]^*$ appears in some subinterval $[a_i^{\alpha}, a_{i+1}^{\alpha}]$ as follows. Formulate as an admissible statement (true in R) the assertion:

"For every $j \in N$, every point of $[a, b]$ is in the subinterval $[a_i^j, a_{i+1}^j]$ for some $i \in I$ where $0 \leqq i < j$."

Putting in appropriate stars *, it becomes true in R^*. Particularizing it to the case where $j = \alpha$ we get:

"Every point of $[a, b]^*$ is in the subinterval $[a_i^{\alpha}, a_{i+1}^{\alpha}]$ for some $i \in I^*$ where $0 \leqq i < \alpha$."

Using the above we can now show that the partition described there has uncountably many points from which it also follows that $\{ \beta \in N^* \mid \beta \leqq \alpha \}$ and N^* are uncountable. We do this by showing a mapping from $\{ a_0^{\alpha}, a_2^{\alpha}, \ldots, a_{\alpha}^{\alpha} \}$ onto the standard interval $[a, b]$ which is known to be uncountable. Each partition point a_i^{α} being finite is infinitely close (Theorem 2.1) to a uniquely determined real. Let the image of a_i^{α} be that real. Clearly the image is in $[a, b]$. Moreover the mapping is onto because we saw that each real c in [a, b] is a member of $[a_i^{\alpha}, a_{i+1}^{\alpha}]$ some $0 \leqq i < \alpha$, and since $a_i^{\alpha} \approx a_{i+1}^{\alpha}$, we must also have $c \approx a_i^{\alpha}$.

Consider the following novel proof of a famous theorem.

THEOREM 8.1. If the standard function f is continuous on the standard interval $[a, b]$ and is negative at a and positive at b, then at some standard point c in the interval, $f(c) = 0$.

PROOF. Let α be an infinite natural number and form the infinitely fine partition $\{ a_0^{\alpha}, a_2^{\alpha}, \ldots, a_{\alpha}^{\alpha} \}$ described earlier in this section. Now the following assertion can be formulated as an admissible statement true in R:

"For each $j \in N$ there exists a least $i \in N$ such that $0 < i \leqq j$ and $f(a_i^j) \geqq 0$."

Putting in stars this becomes true in R^*. Now particularizing it to the case $j = \alpha$ we get (leaving off some stars for brevity):

"Exists least $i \in N^*$ such that $0 < i \leqq \alpha$ and $f(a_i^{\alpha}) \geqq 0$."

For this i then we must have $f(a_{i-1}^{\alpha}) < 0$. Now a_i^{α} is finite and must be infinitely close to a standard number c in the interval. Since f is a standard function, $f(c)$ is standard. Now from $a_{i-1}^{\alpha} \approx a_i^{\alpha}$ we get

$$c \approx a_i^{\alpha} \text{ and } c \approx a_{i-1}^{\alpha}.$$

Then by continuity we see that

$$f(c) \approx f(a_i^\alpha) \text{ and } f(c) \approx f(a_{i-1}^\alpha).$$

Taking this together with the fact (seen already) that

$$f(a_i^\alpha) \geqq 0 \text{ and } f(a_i^\alpha) < 0$$

we have (in summary) that $f(c)$ is a standard number infinitely close to a negative number and a non-negative number. Thus $f(c) = 0$.

(Q.E.D.)

9. Derivatives

Let $f(x)$ be a standard function defined on a standard open interval (a, b) and having the point x_0 as an interior point. Using the non-standard characterization of limit, the condition that $f(x)$ be differentiable at x_0 is that there exist a standard number L such that

$$\frac{f(x_0+dx) - f(x_0)}{dx} \approx L$$

for all non-zero infinitesimals dx. L, of course, will be the derivative. If $f(x)$ is differentiable, then writing $dy = f(x_0+dx) - f(x_0)$ we have (using the notation for "standard part" introduced in Section 2) $°(\frac{dy}{dx}) = f'(x_0)$. This says that the quotient of the infinitesimal increments need not in general be the derivative, but it must be infinitely close to it.

Example 9.1. Suppose we wish to calculate the derivative of $f(x) = x^2$. Let dx be an arbitrary non-zero infinitesimal, then

$$\frac{dy}{dx} = \frac{(x+dx)^2 - x^2}{dx}$$

After squaring and cancelling we get, $\frac{dy}{dx} = 2x + dx \approx 2x$ therefore

$$°(\frac{dy}{dx}) = 2x.$$

That is, the function x^2 is differentiable with derivative $2x$.

Example 9.2. Let's see how to prove the Chain Rule! Suppose $f(x)$ and $g(x)$ are differentiable at the appropriate places and we wish to show that the function $h(x) = f(g(x))$ is differentiable with derivative $h'(x) = f'(g(x))\, g'(x)$. For any non-zero infinitesimal dx, write $dg = g(x+dx) - g(x)$ and $dh = h(x+dx) - h(x)$ then

$$dh = f(g(x + dx)) - f(g(x)) = f(g(x) + dg) - f(g(x)).$$

We want to show that for any non-zero infinitesimal dx,

$$\frac{dh}{dx} \approx f'(g(x))\, g'(x)\,. \tag{1}$$

Let non-zero infinitesimal dx be given. By continuity of $g(x)$, dg is also infinitesimal.

Case 1. $dg = 0$. Then $dh = 0$, so $°\left(\frac{dg}{dx}\right) = g'(x) = 0$ and $\frac{dh}{dx} = 0$. Thus both sides of (1) are zero, so (1) holds.

Case 2. $dg \neq 0$. Then $\frac{dh}{dx} = \frac{dh}{dg} \cdot \frac{dg}{dx}$ that is

$$\frac{dh}{dx} = \frac{f(g(x)+dg) - f(g(x))}{dg} \cdot \frac{g(x+dx) - g(x)}{dx}\,. \tag{2}$$

The two factors of the right side of (2) are infinitely close to $f'(g(x))$ and $g'(x)$ respectively. Now using the rules given in Section 2 for manipulating the symbol $\approx$ we get

$$\frac{dh}{dx} \approx f'(g(x)) \cdot g'(x)$$

as desired.

10. Integration

Let $f(x)$ be a standard function integrable on the standard interval $[a, b]$. For each standard n let

$$a = a_0^n < a_1^n < \cdots < a_n^n = b$$

be a partition of the interval into n subintervals of equal length. The Riemann sums

$$S_n = \sum_{i=1}^{n} f(a_i^n)\,(a_i^n - a_{i-1}^n)$$

constitute an infinite sequence, and by the Main Theorem this sequence can be extended to a sequence defined on N^*. For an infinite natural number α it seems natural to denote the α^{th} term S_α by

$$\sum_{i=1}^{\alpha} f(a_i^\alpha)\,(a_i^\alpha - a_{i-1}^\alpha)\,. \tag{1}$$

We might think of this as a "Riemann sum" on an infinitely fine net. The use of $\sum$ notation seems appropriate because the "sum" shares (by virtue of the Main Theorem) many properties of standard finite sums. For example, the property (omitting the summands for brevity)

$$\sum_{i=1}^{\alpha} = \sum_{i=1}^{\beta} + \sum_{i=\beta+1}^{\alpha}\,.$$

Now since

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(a_i^n)\,(a_i^n - a_{i-1}^n) = \int_a^b f(x)\,dx\,,$$

we see, using the non-standard characterization of the notion of limit of a sequence, that if α is an infinite natural number

$$\sum_{i=1}^{\alpha} f(a_i^\alpha)\,(a_i^\alpha - a_{i-1}^\alpha) \approx \int_a^b f(x)\,dx\,.$$

A further development of the theory of Integration and in particular a non-standard characterization of the Riemann integrable functions requires more machinery than we are prepared to set up here.

11. The Main Theorem Revisited

The version of the Main Theorem which we gave you in Section 3 is a specialization of a considerably more general result. While we stated it in terms of the real number system R, it happens to be true of any non-empty set X whatsoever. This opens the way for a penetration of the methods of Non-Standard Analysis to other branches of mathematics. For example, one might extend the complex number system C to a field C^*. There one could have "polygons" with sides of infinitely small length and vertices indexed by the initial segment of N^* determined by some infinite natural number.

What is also true is that R^* is even more closely related to R than we've suggested so far. Our version of the Main Theorem didn't permit as admissible statements those which had variables ranging over the functions on R, the relations on R or the subsets of R. A generalization of the theorem to include such statements is impossible. If it were possible, the Axiom of Completeness would be admissible and we'd have that R^* is complete, which contradicts a result seen previously. It turns out, however, that there exists a distinguished class of functions on R^*, a distinguished class of relations on R^*, and a distinguished class of subsets of R^* such that all statements with function, relation, and set variables can now be allowed in applications of the theorem provided that in R^* these variables are constrained to vary only over these distinguished classes. Robinson calls the functions, relations, and subsets of R^* in these classes *internal* functions, relations and subsets. Expressed differently what we are saying is that if you wore spectacles which were opaque to all functions, relations, and subsets of R^* other than the internal ones, you'd swear that R^* is complete, N^* is well ordered, etc. Your glasses wouldn't let you see the counterexamples! What is remarkable is that *one* pair of spectacles can be made to work for *all* the new statements. If only the Axiom of Completeness were at issue, we could simply choose a pair of spectacles which blocks out the bounded subsets of R^* which don't have least upper bounds. Using the improvement of the Main Theorem just mentioned the Theory of Integration, for example, becomes more susceptible to the methods of Non-Standard Analysis, and some of the argumentation elsewhere in this article could be simplified.

Conclusion

At the turn of the century Bertrand Russell wrote:

> "... hence infinitesimals as explaining continuity must be regarded as unnecessary, erroneous, and self contradictory."

This remark gives some indication of the degree of disrepute into which the use of infinitesimals had fallen, and it serves to underscore the achievement in its eventual vindication by Robinson. Russell's work in logic, it should be mentioned, constituted one of the important steps along the way. Such is the unexpected path the development of ideas sometimes follows!

Exercises

Functions, relations, and subsets are assumed to be standard.

1. Prove that a continuous one-one function on a compact set has a continuous inverse.

2. Prove that a continuous image of a compact set is compact.

3. Give a non-standard characterization of:
   (a) "The set S is open."
   (b) "Point p (standard) is a limit point of the set T."

4. Prove that a set is compact if and only if it is closed and bounded.

5. Show that S is a proper subset of S^* if and only if S is infinite.

6. Show that if $\lim_{n \to 0} a_n = 0$, then $\lim_{n \to 0} \frac{a_1 + a_2 + \cdots + a_n}{n} = 0$.

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