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The discriminant is $4p^2(p-\alpha^2 r^2)$ which cannot vanish, so that, as before, the first factor in (10) must be zero, and we have

$$(\alpha^2+\beta^2)\,r^2 - p = 0 \tag{11}$$

which is a contradiction since $\alpha^2 + \beta^2 > 1$ and we are supposing that $|r| > 1$.

Therefore we cannot have $|r| > 1, K \neq 0$, and $L \neq 0$. If $|r| = 1$ we see that $K = L = 0$ since $|x - \alpha r| < p$ and $|y - \beta r| < p$ in this case. If $|r| > 1$ with $K = L = 0$ we would have $x = \alpha r$, $y = \beta r$ and hence $(x, y) > 1$, whereas x and y are relatively prime. Finally it remains to consider the possibility of having $|r| > 1$ with one of K and L zero, the other nonzero. This if we suppose that $|r| > 1$, $K = 0$, $L \neq 0$, we obtain (9) which, as we have seen, leads to a contradiction. On the other hand the supposition that $|r| > 1$ with $K \neq 0$, $L = 0$ implies that (11) would hold with $r^2 > 1$.

We conclude that $|r| = 1$, $K = 0$ and $L = 0$. Hence $x = \pm\alpha$, $y = \pm\beta$ and $\alpha^2 + \beta^2 = p$.

In [1], Corollary 2, we observed that if $p = x^2 + y^2$ then, in our notation, y is a quadratic residue of p. Collecting our results we have the

COROLLARY. Let $p = x^2 + y^2$ where p is a prime of the form $4n + 1$ with x and y given by (3) and (4). Then $\left(\frac{x}{p}\right) = \left(\frac{2}{p}\right)$ and $\left(\frac{y}{p}\right) = 1$.

5. Conclusion

We saw that $x = \pm\alpha$, $y = \pm\beta$. When $p = 13$ we have $y = -3$, $\beta = -3$; when $p = 29$, $y = -5$, $\beta = 5$, and when $p = 41$, $y = 5$, $\beta = 5$. Hence the sign of y, determined by the approximants to a continued fraction depends on the integer m, the number of terms in the finite segment of (2) which is used, can agree with that of β or be opposite that of β. The same applies to x and α. In [1], Theorem 1, we gave a construction which always gives positive values for x and y. Other various constructions, as we have seen, do not have this property.

Finally we comment on the numbers $\dfrac{(2n)!}{2(n!)^2}$ which we denote by a_n for $n = 1, 2, 3, \ldots$.

The members of the sequence $\{ a_n \}$ are related to the numbers $b_{n+1} = \frac{(2n)!}{(n+1)!\,n!}$, $n = 0, 1, 2, \ldots$, which, as mentioned by Becker [2], have a variety of applications. Birkhoff [3] pointed out that b_n is an integer for every positive integer n, and noted the recurrence relation $b_n = \sum_{i=1}^{n-1} b_i b_{n-i}$; a relation which was also obtained by Wedderburn [10].

The results of this note depend on the fact that a_n is an integer, at least when $p = 4n + 1$ is a prime. Although it is known that a_n is an integer for every positive integer n, we can see that this also follows readily from [3]. For we have $2a_n = (n+1)\,b_{n+1}$. If n is even, it follows that b_{n+1} is even since $(2, n+1) = 1$. Therefore $a_n = (n+1)\frac{b_{n+1}}{2}$ is an integer. If n is odd then $2 \mid (n+1)$ and in this case also $a_n = \frac{n+1}{2}\, b_{n+1}$ is an integer. A list of values for a_n can be obtained from the second column of a table in [2], page 699, headed N_n, by multiplying the $(n+1)$st member by $\frac{n+1}{2}$.

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