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pect to an odd prime. We finally get $(-1)^{n+1} \equiv (-1)^{2n^2+n}$ or $(-1)^{n+1} \equiv (-1)^n \pmod{p}$ which is a contradiction since p is an odd prime. Thus $\frac{(2n)!}{2(n!)^2}$ is a quadratic residue of p.

4. The Construction of Gauss

THEOREM. Suppose p = 4n + 1 is a prime and $p = x^2 + y^2$ where x and y are given by (3) and (4). Let β and α denote respectively the numerically smallest residues of $\frac{(2n)!}{2(n!)^2}$ and $(2n)!\beta$ modulo p, so that $|\alpha| < \frac{p}{2}$, $|\beta| < \frac{p}{2}$. Then $p = \alpha^2 + \beta^2$. Proof. By (5) we have, using the remark at the beginning of section 3, $A^2 + A^2 = 0 \pmod{p}$ and hence $A^2 = A^2 \pmod{p}$.

A_m² + A_{m-1}² $\equiv 0 \pmod{p}$ and hence $-A_m^2 \equiv A_{m-1}^2 \pmod{p}$, so that $\{(2n) !\}^2 A_m^2 \equiv A_{m-1}^2 \pmod{p}$, and since p is a prime $(2n) !A_m \equiv \pm A_{m-1} \pmod{p}$. Supposing the negative sign holds we have $(2n) !A_m^2 \equiv -A_m A_{m-1} \pmod{p}$. Therefore we obtain $(2n) !A_m^2 - (2n) !PB_m^2 \equiv (pB_mB_{m-1} - A_mA_{m-1}) \pmod{p}$, so that by (3) and (4) we get

(6)
$$x \equiv (2n) \mid y \pmod{p}$$

If the positive sign holds above it follows that $x \equiv -(2n) ! y \pmod{p}$ which is just as good for our present purposes since we are not concerned with the signs of x and y. We will comment on the signs in section 5.

By the lemma we have $\left\{\frac{(2n)!}{2(n!)^2}\right\}^{\frac{p-1}{2}} \equiv 1 \pmod{p}$ so $(2n)!^{\frac{p-1}{2}} \equiv 2^{\frac{p-1}{2}}$ $(n!)^{p-1} \pmod{p}$, and therefore $(2n)!^{\frac{p-1}{2}} \equiv 2^{\frac{p-1}{2}} \pmod{p}$ since (n!, p) = 1. We have $x \equiv \pm (2n)! y \pmod{p}$, and since each of y and -1 is a quadratic residue of p, $x^{\frac{p-1}{2}} \equiv (2n)!^{\frac{p-1}{2}} \equiv 2^{\frac{p-1}{2}} \pmod{p}$, and in terms of the Legendre symbol it follows that $\left(\frac{x}{p}\right) = \left(\frac{2}{p}\right)$; that is, the quadratic character of 2 with respect to p.

Suppose 2 is a quadratic residue of p. Then

$$2^{\frac{p-1}{2}}(n !)^{p-1}(A_m A_m - 1)^{\frac{p-1}{2}} \equiv (A_m A_{m-1})^{\frac{p-1}{2}} \equiv (-x)^{\frac{p-1}{2}} \equiv x^{\frac{p-1}{2}} \equiv 1$$

(mod p).

Next, if 2 is a quadratic nonresidue of p we have

 $2^{\frac{p-1}{2}}(n!)^{p-1}(A_mA_{m-1})^{\frac{p-1}{2}} \equiv -(-x)^{\frac{p-1}{2}} \equiv -(x)^{\frac{p-1}{2}} \equiv -(-1) \equiv 1$ (mod p),

and we conclude that $2(n !)^2 A_m A_{m-1}$ is a quadratic residue of p. By (3), (4), and (6) we have

$$(2n) ! y \equiv -A_m A_{m-1} \pmod{p},$$

$$2(n !)^2 (2n) ! y \equiv -2(n !)^2 A_m A_{m-1} \pmod{p}$$

and

$$(-2(n !)^2 (2n) ! y \equiv b^2 \pmod{p}$$

for some quadratic residue
$$b^2$$
. Therefore

$$-2(n !)^{2} (2n) ! y \equiv -(2n) !^{2} b^{2} \pmod{p},$$

$$-2(n !)^{2} y \equiv -(2n) ! b^{2} \pmod{p},$$

and finally

$$y \equiv \frac{(2n)!}{2(n!)^2} b^2 \pmod{p}$$
.

Hence by (6)

$$x \equiv \frac{(2n)!^2}{2(n!)} b^2 \pmod{p}.$$

Let $b^2 \equiv r \pmod{p}$, $|r| < \frac{p}{2}$, so that (r, p) = 1. Then in terms of α ,

 β , and r, $x \equiv \alpha r \pmod{p}$ and $y \equiv \beta r \pmod{p}$. There are unique integers K and L such that $x = \alpha r + Kp$, $y = \beta r + Lp$. Then

$$x^{2} + y^{2} = (\alpha^{2} + \beta^{2}) r^{2} + (K^{2} + L^{2}) p^{2} + 2rp(\alpha K + \beta L),$$

or

$$p = (\alpha^2 + \beta^2) r^2 + (K^2 + L^2) p^2 + 2rp(\alpha K + \beta L).$$

Suppose that $|r| > 1, K \neq 0$, and $L \neq 0$. The last equation can be written

(7)
$$pK^2 + (2r\alpha p)K + \{L^2p^2 + 2r\beta pL + (\alpha^2 + \beta^2)r^2 - p\} = 0.$$

Since (7) is a quadratic in K and we are supposing that the integral root is not zero we have

$$K | \{ L^2 p^2 + 2r\beta pL + (\alpha^2 + \beta^2) r^2 - p \}.$$

There is an integer t such that

$$L^{2}p^{2} + 2r\beta pL + (\alpha^{2} + \beta^{2})r^{2} - p = Kt$$

and therefore (7) vanishes when

$$K = \frac{L^2 p^2 + 2r\beta p L + (\alpha^2 + \beta^2) r^2 - p}{t}$$

That is

(8)
$$\{ L^2 p^2 + 2r\beta pL + (\alpha^2 + \beta^2) r^2 - p \} \{ t^2 + 2r\alpha pt + p \{ L^2 p^2 + 2r\beta pL + (\alpha^2 + \beta^2) r^2 - p \} \} = 0$$

The discriminant of the quadratic function

$$t^{2} + 2r\alpha pt + p \{ p^{2}L^{2} + 2r\beta pL + (\alpha^{2} + \beta^{2})r^{2} - p \}$$

is $4p^2 \{ p - (pL + \beta r)^2 \}$ which is not zero. It follows that the second factor in (8) cannot be zero; otherwise we would have two distinct integral values for t giving rise to two distinct integers K, whereas K is unique. Hence we have

(9)
$$p^{2}L^{2} + 2r\beta pL + (\alpha^{2} + \beta^{2})r^{2} - p = 0$$

and since we are supposing that $L \neq 0$, we see that

 $L \mid \{ (\alpha^2 + \beta^2) r^2 - p \}$ so that for an integer *u* we have $(\alpha^2 + \beta^2) r^2 - p = L u$ and (9) vanishes when

$$L = \frac{\left(\alpha^2 + \beta^2\right)r^2 - p}{u},$$

so that

(10)
$$\{(\alpha^2 + \beta^2)r^2 - p\}\{u^2 + 2r\beta pu + p^2\{(\alpha^2 + \beta^2)r^2 - p\}\} = 0.$$

As before we consider the quadratic function

 $u^{2} + 2r\beta pu + p^{2} \{ (\alpha^{2} + \beta^{2}) r^{2} - p \}$

The discriminant is $4p^2(p-\alpha^2r^2)$ which cannot vanish, so that, as before, the first factor in (10) must be zero, and we have

(11)
$$(\alpha^2 + \beta^2) r^2 - p = 0$$

which is a contradiction since $\alpha^2 + \beta^2 > 1$ and we are supposing that |r| > 1.

Therefore we cannot have $|r| > 1, K \neq 0$, and $L \neq 0$. If |r| = 1 we see that K = L = 0 since $|x - \alpha r| < p$ and $|y - \beta r| < p$ in this case. If |r| > 1 with K = L = 0 we would have $x = \alpha r, y = \beta r$ and hence (x, y) > 1, whereas x and y are relatively prime. Finally it remains to consider the possibility of having |r| > 1 with one of K and L zero, the other nonzero. This if we suppose that $|r| > 1, K = 0, L \neq 0$, we obtain (9) which, as we have seen, leads to a contradiction. On the other hand the supposition that |r| > 1 with $K \neq 0, L = 0$ implies that (11) would hold with $r^2 > 1$.

We conclude that |r| = 1, K = 0 and L = 0. Hence $x = \pm \alpha$, $y = \pm \beta$ and $\alpha^2 + \beta^2 = p$.

In [1], Corollary 2, we observed that if $p = x^2 + y^2$ then, in our notation, y is a quadratic residue of p. Collecting our results we have the

COROLLARY. Let $p = x^2 + y^2$ where p is a prime of the form 4n + 1with x and y given by (3) and (4). Then $\left(\frac{x}{p}\right) = \left(\frac{2}{p}\right)$ and $\left(\frac{y}{p}\right) = 1$.

5. CONCLUSION

We saw that $x = \pm \alpha$, $y = \pm \beta$. When p = 13 we have y = -3, $\beta = -3$; when p = 29, y = -5, $\beta = 5$, and when p = 41, y = 5, $\beta = 5$. Hence the sign of y, determined by the approximants to a continued fraction depends on the integer m, the number of terms in the finite segment of (2) which is used, can agree with that of β or be opposite that of β . The same applies to x and α . In [1], Theorem 1, we gave a construction which always gives positive values for x and y. Other various constructions, as we have seen, do not have this property.

Finally we comment on the numbers $\frac{(2n)!}{2(n!)^2}$ which we denote by a_n for n = 1, 2, 3, ...