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**Autor:** Taylor, D. E.  
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## 2. DIVISION RINGS

By *division ring* we mean an associative ring with identity in which every non-zero element has an inverse. If  $D$  is a division ring, the *normalizer*  $N(F)$  of a subfield  $F$  consists of those elements  $d$  such that  $dF = Fd$ , while the *centralizer*  $C(F)$  consists of those elements  $d$  such that  $dx = xd$  for all  $x$  in  $F$ ; the centralizer is a subdivision ring of  $D$ .

From now on  $D$  will denote a division ring with centre  $K$  and  $F$  will denote a maximal subfield of  $D$ . We shall assume that  $F = K(\theta)$  where  $\theta$  satisfies an irreducible monic polynomial  $f$  with coefficients in  $K$  which splits into distinct linear factors over  $F$ . We shall see below that this assumption allows us to apply the results of §1 to  $D$  considered as a vector space over  $F$  (multiplying on the left with elements of  $F$ ). For each element  $a$  of  $D$ , the assignment  $T_a(d) = da$  defines a linear transformation  $T_a$  of this vector space.

If  $d$  is an eigenvector of  $T_\theta$ , then for some  $\lambda$  in  $F$ ,  $d\theta = \lambda d$ . This implies that  $d\theta d^{-1} = \lambda$  and hence  $dFd^{-1} = F$ ; thus  $d \in N(F)$ . Conversely, if  $d \in N(F)$  and  $d \neq 0$ , then  $d\theta d^{-1} = \lambda \in F$  for some  $\lambda$  and hence  $d$  is an eigenvector of  $T_\theta$ . This proves

(2.1) *A non-zero element  $d$  of  $D$  is an eigenvector of  $T_\theta$  if and only if it belongs to  $N(F)$ .*

Since  $f(T_\theta) = 0$ , the conditions of §1 apply and we have

(2.2) *The vector space  $D$  is the direct sum of the eigenspaces of  $T_\theta$ .*

Let  $\lambda$  be an eigenvalue of  $T_\theta$  with eigenvector  $d$ , then as above  $d\theta = \lambda d$ . If  $d'$  is another eigenvector, then  $d'd^{-1}\lambda d d'^{-1} = \lambda$  and  $d'd^{-1}$  centralizes  $F$  since  $F = K(\lambda)$ . However,  $F$  is a maximal subfield, and therefore self-centralizing, so  $d' = ed$  for some  $e$  in  $F$ . Thus we obtain

(2.3) *Each eigenspace of  $T_\theta$  has dimension one.*

Next, we wish to show that  $f(X)$  is the minimal polynomial of  $T_\theta$ . Let  $\theta = \theta_1, \theta_2, \dots, \theta_m$  be the eigenvalues of  $T_\theta$  and let  $1 = d_1, d_2, \dots, d_m$  be corresponding eigenvectors. Because  $N(F)$  is multiplicatively closed  $d_i d_j$  must correspond to an eigenvalue  $\theta_k$ , say, and hence  $d_i d_j \theta = \theta_k d_i d_j$ , which implies that  $d_i \theta_j = \theta_k d_i$ . This shows that the mapping which takes  $\theta_j$  to  $d_i \theta_j d_i^{-1}$  permutes the eigenvalues among themselves. Consequently, the coefficients of  $g(X) = (X - \theta_1) \dots (X - \theta_m)$  commute with all the eigen-

vectors and they therefore belong to the centre of  $D$  since the eigenvectors span  $D$ . Each eigenvalue is a root of  $f(X)$  so the degree of  $g(X)$  is no larger than that of  $f(X)$ . But  $g(\theta) = 0$  so we must have  $g(X) = f(X)$ . Since each  $\theta_i$  must be a root of the minimal polynomial of  $T_\theta$  this proves

(2.4) *The minimal polynomial of  $T_\theta$  is  $f(X)$ .*

As immediate consequences we have

(2.5)  $\dim_F D = \dim_K F = \text{degree of } f = m.$

(2.6)  $\dim_K D = m^2.$

Finally, we prove

(2.7) *If  $E = K(\theta')$  and  $f(\theta') = 0$ , then for some non-zero element  $d$  of  $D$ ,  $d E d^{-1} \subseteq F$ .*

To see this, consider the linear transformation  $T_{\theta'}$ . Since  $f(T_{\theta'}) = 0$  there is an eigenvalue  $\lambda \in F$  of  $T_{\theta'}$  and a corresponding eigenvector  $d$  such that  $d \theta' = \lambda d$ ; it follows that  $d E d^{-1} \subseteq F$ .

*Remark.* The assumption on the field  $F$  amounts to supposing that  $F/K$  is a finite Galois extension and the proof of (2.4) shows that  $N(F)^\# / F^\#$  is isomorphic to its Galois group. (Where  $F^\#$  denotes the set of non-zero elements of  $F$ .)

### 3. WEDDERBURN'S THEOREM

This proof follows van der Waerden [14, p. 203]. The counting argument was used by Artin [1] in his proof of the same theorem.

**THEOREM.** *Every finite division ring is a field.*

*Proof.* Suppose that  $D$  is a finite division ring with centre  $K$  and maximal subfield  $F$ . If the order of  $F$  is  $q$ , then the elements of  $F$  constitute all the roots of the polynomial  $X^q - X$ ; hence any two finite fields of the same order are isomorphic. The multiplicative group of a finite field is cyclic, so  $F = K(\theta)$  for some  $\theta$ . Any element of  $D$  is contained in a maximal subfield, which by (2.5) has the same order as  $F$  and hence by (2.7) any element of the multiplicative group  $G$  of non-zero elements of  $D$  belongs to a conjugate of  $H$ , the multiplicative group of non-zero elements of  $F$ . The