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Autor: Taylor, D. E.
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SOME CLASSICAL THEOREMS ON DIVISION RINGS

by D. E. TAYLOR

The theorem of Wedderburn [15] that every finite division ring is a field, and the theorem of Frobenius [6] characterizing the quaternions as a non-commutative real division algebra can both be obtained as immediate and easy consequences of theorems on central simple algebras—particularly the Skolem-Noether theorem (van der Waerden [14, p. 199]). The purpose of this note is to use elementary linear algebra to prove a version of the Skolem-Noether theorem sufficient to yield the results of Wedderburn and Frobenius.

1. SOME LINEAR ALGEBRA

All the results of this section are quite elementary and can be found in most texts on linear algebra (for example: Hoffman and Kunze [9]).

Let V be a vector space over a field F and let T be a linear transformation of V. Suppose that $f(X)$ is a polynomial with coefficients in F such that $f(T) = O$. If $f(X) = f_1(X) f_2(X)$ where $f_1(X)$ and $f_2(X)$ are coprime, then there are polynomials $g_1(X)$ and $g_2(X)$ such that $1 = f_1(X) g_1(X) + f_2(X) g_2(X)$. Then for each v in V the vector $v_1 = f_2(T) g_2(T) v$ belongs to the kernel, V_1, of $f_1(T)$, the vector $v_2 = f_1(T) g_1(T) v$ belongs to the kernel, V_2, of $f_2(T)$ and $v = v_1 + v_2$. Thus V is the (direct) sum of V_1 and V_2. Moreover, the restriction T_i of T to V_i satisfied the equation $f_i(T_i) = 0$ for $i = 1, 2$.

It follows by induction on the degree that if $f(X)$ can be factorized over F into distinct linear factors, then V is the direct sum of the eigenspaces of T. Note that V is not assumed to be finite dimensional.

Recall that the minimal polynomial of T is the monic polynomial $m(X)$ of least degree such that $m(T) = 0$. It is immediate that each eigenvalue λ of T satisfies the equation $m(\lambda) = 0$ and conversely, the above considerations show that each root of $m(X)$ is an eigenvalue of T.