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SOME CLASSICAL THEOREMS ON DIVISION RINGS

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The theorem of Wedderburn [15] that every finite division ring is a field, and the theorem of Frobenius [6] characterizing the quaternions as a non-commutative real division algebra can both be obtained as immediate and easy consequences of theorems on central simple algebras—particularly the Skolem-Noether theorem (van der Waerden [14, p. 199]). The purpose of this note is to use elementary linear algebra to prove a version of the Skolem-Noether theorem sufficient to yield the results of Wedderburn and Frobenius.

1. SOME LINEAR ALGEBRA

All the results of this section are quite elementary and can be found in most texts on linear algebra (for example: Hoffman and Kunze [9]).

Let V be a vector space over a field F and let T be a linear transformation of V. Suppose that $f(X)$ is a polynomial with coefficients in F such that $f(T) = O$. If $f(X) = f_1(X) f_2(X)$ where $f_1(X)$ and $f_2(X)$ are coprime, then there are polynomials $g_1(X)$ and $g_2(X)$ such that $1 = f_1(X) g_1(X) + f_2(X) g_2(X)$. Then for each v in V the vector $v_1 = f_2(T) g_2(T) v$ belongs to the kernel, V_1, of $f_1(T)$, the vector $v_2 = f_1(T) g_1(T) v$ belongs to the kernel, V_2, of $f_2(T)$ and $v = v_1 + v_2$. Thus V is the (direct) sum of V_1 and V_2. Moreover, the restriction T_i of T to V_i satisfied the equation $f_i(T_i) = 0$ for $i = 1, 2$.

It follows by induction on the degree that if $f(X)$ can be factorized over F into distinct linear factors, then V is the direct sum of the eigenspaces of T. Note that V is not assumed to be finite dimensional.

Recall that the minimal polynomial of T is the monic polynomial $m(X)$ of least degree such that $m(T) = 0$. It is immediate that each eigenvalue λ of T satisfies the equation $m(\lambda) = 0$ and conversely, the above considerations show that each root of $m(X)$ is an eigenvalue of T.

2. Division rings

By *division ring* we mean an associative ring with identity in which every non-zero element has an inverse. If D is a division ring, the *normalizer* $N(F)$ of a subfield F consists of those elements d such that $dF = Fd$, while the *centralizer* $C(F)$ consists of those elements d such that $dx = xd$ for all x in F; the centralizer is a subdivision ring of D.

From now on D will denote a division ring with centre K and F will denote a maximal subfield of D. We shall assume that $F = K(\theta)$ where θ satisfies an irreducible monic polynomial f with coefficients in K which splits into distinct linear factors over F. We shall see below that this assumption allows us to apply the results of §1 to D considered as a vector space over F (multiplying on the left with elements of F). For each element a of D, the assignment $T_a(d) = da$ defines a linear transformation T_a of this vector space.

If d is an eigenvector of T_θ, then for some λ in F, $d\theta = \lambda d$. This implies that $d\theta d^{-1} = \lambda$ and hence $dFd^{-1} = F$; thus $d \in N(F)$. Conversely, if $d \in N(F)$ and $d \neq 0$, then $d\theta d^{-1} = \lambda \in F$ for some λ and hence d is an eigenvector of T_θ. This proves

(2.1) *A non-zero element d of D is an eigenvector of T_θ if and only if it belongs to $N(F)$.*

Since $f(T_\theta) = 0$, the conditions of §1 apply and we have

(2.2) *The vector space D is the direct sum of the eigenspaces of T_θ.*

Let λ be an eigenvalue of T_θ with eigenvector d, then as above $d\theta = \lambda d$. If d' is another eigenvector, then $d'd^{-1}\lambda d d'^{-1} = \lambda$ and $d'd^{-1}$ centralizes F since $F = K(\lambda)$. However, F is a maximal subfield, and therefore self-centralizing, so $d' = ed$ for some e in F. Thus we obtain

(2.3) *Each eigenspace of T_θ has dimension one.*

Next, we wish to show that $f(X)$ is the minimal polynomial of T_θ. Let $\theta = \theta_1, \theta_2, \ldots, \theta_m$ be the eigenvalues of T_θ and let $1 = d_1, d_2, \ldots, d_m$ be corresponding eigenvectors. Because $N(F)$ is multiplicatively closed $d_i d_j$ must correspond to an eigenvalue θ_k, say, and hence $d_i d_j \theta = \theta_k d_i d_j$, which implies that $d_i \theta_j = \theta_k d_i$. This shows that the mapping which takes θ_j to $d_i \theta_j d_i^{-1}$ permutes the eigenvalues among themselves. Consequently, the coefficients of $g(X) = (X - \theta_1) \ldots (X - \theta_m)$ commute with all the eigenvectors and they therefore belong to the centre of D since the eigenvectors span D. Each eigenvalue is a root of $f(X)$ so the degree of $g(X)$ is no larger than that of $f(X)$. But $g(\theta) = 0$ so we must have $g(X) = f(X)$. Since each θ_i must be a root of the minimal polynomial of T_θ this proves

(2.4) *The minimal polynomial of* T_θ *is* $f(X)$.

As immediate consequences we have

(2.5) $\dim_F D = \dim_K F =$ degree of $f = m$.

(2.6) $\dim_K D = m^2$.

Finally, we prove

(2.7) *If* $E = K(\theta')$ *and* $f(\theta') = 0$, *then for some non-zero element* d *of* D, $d E d^{-1} \subseteq F$.

To see this, consider the linear transformation $T_{\theta'}$. Since $f(T_{\theta'}) = 0$ there is an eigenvalue $\lambda \in F$ of $T_{\theta'}$ and a corresponding eigenvector d such that $d\theta' = \lambda d$; it follows that $d E d^{-1} \subseteq F$.

Remark. The assumption on the field F amounts to supposing that F/K is a finite Galois extension and the proof of (2.4) shows that $N(F)^{\#}/F^{\#}$ is isomorphic to its Galois group. (Where $F^{\#}$ denotes the set of non-zero elements of F.)

3. Wedderburn's theorem

This proof follows van der Waerden [14, p. 203]. The counting argument was used by Artin [1] in his proof of the same theorem.

Theorem. *Every finite division ring is a field.*

Proof. Suppose that D is a finite division ring with centre K and maximal subfield F. If the order of F is q, then the elements of F constitute all the roots of the polynomial $X^q - X$; hence any two finite fields of the same order are isomorphic. The multiplicative group of a finite field is cyclic, so $F = K(\theta)$ for some θ. Any element of D is contained in a maximal subfield, which by (2.5) has the same order as F and hence by (2.7) any element of the multiplicative group G of non-zero elements of D belongs to a conjugate of H, the multiplicative group of non-zero elements of F. The number of conjugates of a subgroup is the index of its normalizer, so H has at most $|G : H|$ conjugates in G and hence the union of the conjugates contains at most $|G : H|(|H| - 1) + 1 = |G| - |G : H| + 1$ elements. This number is less than $|G|$ except when $G = H$. Hence $D = F$ is a field.

4. Frobenius' theorem

Let $\mathbf{R}$ denote the field of real numbers, $\mathbf{C}$ the field of complex numbers and $\mathbf{H}$ the division ring of quaternions. The following proof makes use of the fundamental theorem that every polynomial with coefficients in $\mathbf{C}$ has a root in $\mathbf{C}$.

THEOREM. *Let D be a division ring which contains the real numbers $\mathbf{R}$ in its centre and suppose that every element of D satisfies a polynomial with coefficients in $\mathbf{R}$. Then D is isomorphic to one of $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$.*

Proof. Suppose that D is not isomorphic to $\mathbf{R}$ or $\mathbf{C}$. It follows that the maximal subfield F of D is isomorphic to $\mathbf{C}$, the centre K of D is isomorphic to $\mathbf{R}$ and $F = K(i)$ where $i^2 = -1$. Let j be an eigenvector of T_i corresponding to the eigenvalue $-i$. Then $ji = -ij$ and j^2 commutes with j and F. From (2.2) and (2.3) the elements 1 and j form an F-basis for D and therefore $j^2 = \alpha$ belongs to K. If $\alpha = \beta^2$ for some $\beta \in K$ then $(j-\beta)(j+\beta) = 0$ and j belongs to K, which is not the case; hence $\alpha = -\beta^2$ for some $\beta \in K$. Replacing j by $j\beta^{-1}$ we obtain a K-basis $1, i, j, ij$ for D such that $i^2 = j^2 = -1$ and $ij = -ji$. That is, D is isomorphic to $\mathbf{H}$.

An almost identical argument shows that if the dimension of D over its centre K is 4 and the characteristic is not 2, then D has a K-basis $1, i, j, ij$ where $i^2 = \alpha$, $j^2 = \beta$ and $ij = -ji$ for some $\alpha, \beta \in K$.

5. Other proofs of Wedderburn's theorem

The original proofs of the theorem of §3 were given first by Wedderburn [15] in 1905 and then by Dickson [5] in the same year; they depend on certain divisibility properties of the integers. The neatest proof along these lines is that of Witt [16]. Elementary proofs which avoid the use of such number theory have been given by Artin [1] and Herstein [7]. And proofs which deduce the theorem using finite group theory have been given by Zassenhaus [17], Brandis [3] and Scott [11, p. 426].

Perhaps the most interesting proofs are those which present the result as a consequence of a more general theory. There are two such proofs in the book of van der Waerden [14]: the first (on p. 203) uses the theory of central simple algebras, the second (sketched on p. 215) relates the theorem to cohomology and the Brauer group (see also, Serre [12, p. 170]). The theorem is also a consequence of the work of Tsen [13] and Chevalley [4]. Further comments on the history of the theorem can be found in an article by Artin [2] and in the book by Herstein [8] where many interesting generalisations are also given. One such generalization is a theorem of Jacobson: a division ring in which $x^{n(x)} = x$ for all x is commutative. Laffey [10] has recently given an elementary proof of this using Wedderburn's theorem and linear algebra similar to that used here. See also [18].

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