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by $[a_0, a_1, \ldots, a_n]$. For $0 \leq m \leq n$ we denote the numerator and denominator of the m^{th} approximant to $[a_0, a_1, \ldots, a_n]$ by A_m and B_m respectively.

If p is a prime of the form $4n + 1$, then

$$\sqrt{p} = [a_0, \overline{a_1, \ldots, a_m, a_m, \ldots, a_1, 2a_0}] \tag{2}$$

in the usual notation for a periodic continued fraction. The symmetric part of the period does not have a central term. In [1] we proved that $p = x^2 + y^2$ where

$$x = pB_mB_{m-1} - A_mA_{m-1} \tag{3}$$

$$y = A_m^2 - pB_m^2 \tag{4}$$

and where $\frac{A_m}{B_m}$ is the m^{th} approximant to (2). We also showed that

$$p = \frac{A_m^2 + A_{m-1}^2}{B_m^2 + B_{m-1}^2}. \tag{5}$$

3. The Quadratic Character of

$$\frac{(2n)!}{2(n!)^2}.$$

It is well known that if p is a prime of the form $4n + 1$ then $\{(\frac{p-1}{2})!\}^2 \equiv -1 \pmod{p}$; that is, $(2n)!^2 \equiv -1 \pmod{p}$. We make use of this in the

Lemma. If $p = 4n + 1$ is a prime then $\frac{(2n)!}{2(n!)^2}$ is a quadratic residue of p.

Proof. We use Euler's criterion. Thus if we suppose that $\frac{(2n)!}{2(n!)^2}$ is a quadratic nonresidue of p we have $\{\frac{(2n)!}{2(n!)^2}\}^{\frac{p-1}{2}} \equiv -1 \pmod{p}$ and thus $\{(2n)!^2\}^{\frac{p-1}{4}} \equiv -\{2(n!)^2\}^{\frac{p-1}{2}} \pmod{p}$. Since $(2n)!^2 \equiv -1 \pmod{p}$ and $n!^{p-1} \equiv 1 \pmod{p}$ we have $(-1)^n \equiv -2^{\frac{p-1}{2}} \pmod{p}$, or $(-1)^{n+1} \equiv (-1)^{\frac{p^2+1}{8}}$, using the standard result for the quadratic character of 2 with respect to an odd prime. We finally get $(-1)^{n+1} \equiv (-1)^{2n^2+n}$ or $(-1)^{n+1} \equiv (-1)^n \pmod{p}$ which is a contradiction since p is an odd prime. Thus $\frac{(2n)!}{2(n!)^2}$ is a quadratic residue of p.

4. The Construction of Gauss

Theorem. Suppose $p = 4n + 1$ is a prime and $p = x^2 + y^2$ where x and y are given by (3) and (4). Let β and α denote respectively the numerically smallest residues of $\frac{(2n)!}{2(n!)^2}$ and $(2n)!\,\beta$ modulo p, so that $|\alpha| < \frac{p}{2}$, $|\beta| < \frac{p}{2}$. Then $p = \alpha^2 + \beta^2$.

Proof. By (5) we have, using the remark at the beginning of section 3, $A_m^2 + A_{m-1}^2 \equiv 0 \pmod{p}$ and hence $-A_m^2 \equiv A_{m-1}^2 \pmod{p}$, so that $\{(2n)!\}^2 A_m^2 \equiv A_{m-1}^2 \pmod{p}$, and since p is a prime $(2n)!\,A_m \equiv \pm A_{m-1} \pmod{p}$. Supposing the negative sign holds we have $(2n)!\,A_m^2 \equiv -A_m A_{m-1} \pmod{p}$. Therefore we obtain $(2n)!\,A_m^2 - (2n)!\,pB_m^2 \equiv (pB_m B_{m-1} - A_m A_{m-1}) \pmod{p}$, so that by (3) and (4) we get

$$x \equiv (2n)!\,y \pmod{p}. \tag{6}$$

If the positive sign holds above it follows that $x \equiv -(2n)!\,y \pmod{p}$ which is just as good for our present purposes since we are not concerned with the signs of x and y. We will comment on the signs in section 5.

By the lemma we have $\left\{\frac{(2n)!}{2(n!)^2}\right\}^{\frac{p-1}{2}} \equiv 1 \pmod{p}$ so $(2n)!^{\frac{p-1}{2}} \equiv 2^{\frac{p-1}{2}} (n!)^{p-1} \pmod{p}$, and therefore $(2n)!^{\frac{p-1}{2}} \equiv 2^{\frac{p-1}{2}} \pmod{p}$ since $(n!, p) = 1$. We have $x \equiv \pm (2n)!\,y \pmod{p}$, and since each of y and -1 is a quadratic residue of p, $x^{\frac{p-1}{2}} \equiv (2n)!^{\frac{p-1}{2}} \equiv 2^{\frac{p-1}{2}} \pmod{p}$, and in terms of the Legendre symbol it follows that $\left(\frac{x}{p}\right) = \left(\frac{2}{p}\right)$; that is, the quadratic character of x with respect to p is the same as the quadratic character of 2 with respect to p.

Suppose 2 is a quadratic residue of p. Then