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Artikel: THE NUMBER OF SOLUTIONS OF THE CONGRUENCE $y^2 \equiv x^4 - a \pmod{p}$
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We shall employ a special case of the function (4.1) due to Jacobi. Let $p = ef + 1$ and β be a primitive e -th root of unity. In (4.1) take $\alpha = \beta^n$ (n -integer). We know ([3] page 395) that if e does not divide n then

$$(4.2) \quad F(\beta^n) F(\beta^{-n}) = (-1)^{nf} \cdot p$$

and if we put

$$(4.3) \quad R(n, m) = \frac{F(\beta^n) F(\beta^m)}{F(\beta^{m+n})},$$

then

$$(4.4) \quad R(n, m) = \sum_{h=0}^{e-1} \beta^{nh} \sum_{k=0}^{e-1} \beta^{-(m+n)k} (h, k).$$

From (4.2) and (4.3) it follows that if e does not divide m, n and $m + n$ then

$$(4.5) \quad R(n, m) R(-n, -m) = p$$

and from (4.4) it follows that $R(-n, -m)$ is got from $R(n, m)$ by replacing β by β^{-1} . Let now $e = 4$ and $\beta = \sqrt{-1}$; using (4.4) we get

$$R(1, 1) = (A - B - C - D + 2E) + i(2D - 2B) \text{ in case } p \equiv 1 \pmod{8}$$

$$R(1, 1) = (L - M - R - N + 2S) + i(2M - 2R) \text{ in case } p \equiv 5 \pmod{8}.$$

5. PROOF COMPLETED

If $p \equiv 1 \pmod{4}$ then p splits in $Z[i]$ as $p = \pi \bar{\pi}$ where π is prime in $Z[i]$.

Case 1: $p \equiv 1 \pmod{8}$.

$$\begin{aligned} \sum_{\text{all } v} \chi(v) \chi(v+1) &= \sum_{v \in A_0, A_1, A_2, A_3} \chi(v) \chi(v+1) = \sum_{A_0} + \sum_{A_1} + \sum_{A_2} + \sum_{A_3} \\ &= 1[A + Di - C - Bi] + i[D + Bi - E - Ei] \\ &\quad - 1[C + Ei - C - Ei] - i[B + Ei - E - Di] \\ &= [A - B - C - D + 2E] + i[2D - 2B] \\ (5.1) \quad &= R(1, 1) = (-2f + 8(1, 2) - 1) - 2i[D - B] \end{aligned}$$

where $R(1, 1) \equiv -1 \pmod{2(1+i)}$ (as $D - B = f - 2B - 2E$ by (3.5))

$$\begin{aligned} \text{and } \sum_{\text{all } v} \chi^2(v) \chi^2(v+1) &= (A - D + C - B) - (D - B + E - E) \\ &\quad + (C - E + C - E) - (B - E + E - D) \\ &= A + 3C - D - 2E - B = -1, \end{aligned}$$

by (3.4)

Therefore we have

$$S = \left(\frac{-a}{p}\right) [\chi(-4a) R(1, 1) + \bar{\chi}(-4a) \overline{R(1, 1)} + \chi^2(-4a)(-1)].$$

We put $\pi = -R(1, 1)$, then $\pi \equiv +1 \pmod{2(1+i)}$.

$$\text{Therefore } S = \left(\frac{-a}{p}\right) [-\chi(-4a)\pi - \bar{\chi}(-4a)\bar{\pi} - \chi^2(-4a)].$$

Let g be the primitive root mod p which we have already fixed in § 2. We now have two possibilities:

$$(i) \quad \left(\frac{g}{\pi}\right)_4 = i \quad \text{i.e.} \quad \left(\frac{d}{\pi}\right)_4 = \chi(d) \text{ for all } d.$$

$$(ii) \quad \left(\frac{g}{\pi}\right)_4 = -i \quad \text{i.e.} \quad \left(\frac{d}{\pi}\right)_4 = \bar{\chi}(d) \text{ for all } d.$$

Let $\chi(d) = \left(\frac{d}{\pi^*}\right)_4$ where $\pi^* = \pi$ or $\bar{\pi}$.

We shall show that $\pi^* = \pi$. We have

$$\begin{aligned} \pi &= -\sum \chi(v) \chi(v+1) \equiv -\sum_0^{p-1} v^{\frac{1}{4}(p-1)} (v+1)^{\frac{1}{4}(p-1)} \pmod{\pi^*} \\ &\equiv -[v^{\frac{1}{4}(p-1)} (1 + \frac{1}{4}(p-1)v + \dots + v^{\frac{1}{4}(p-1)})] \pmod{\pi^*}. \end{aligned}$$

In the last sum each term is divisible by $p = \pi \bar{\pi}$, because we know that $\sum_v v^k \equiv 0 \pmod{p}$ unless $p-1/k$. Hence the right hand side of the above is congruent to zero mod π^* . Hence $\pi \equiv 0 \pmod{\pi^*}$ giving $\pi = \pi^*$.

Therefore

$$\begin{aligned} S &= -\left(\frac{-a}{p}\right) \left(\frac{-4a}{\pi}\right)_4 \pi - \left(\frac{-a}{p}\right) \left(\frac{-4a}{\bar{\pi}}\right)_4 \bar{\pi} - \left(\frac{-a}{p}\right) \left(\frac{-4a}{\pi}\right)_4^2 \\ &= -\left(\frac{-4a}{\pi}\right)_4^3 \pi - \left(\frac{-4a}{\bar{\pi}}\right)_4^3 \bar{\pi} - 1 = -\left(\frac{a}{\pi}\right)_4^3 \pi - \left(\frac{a}{\bar{\pi}}\right)_4^3 \bar{\pi} - 1 \\ &= -\left(\frac{a}{\pi}\right)_4 \pi - \left(\frac{a}{\bar{\pi}}\right)_4 \bar{\pi} - 1, \end{aligned}$$

using (i) $\left(\frac{d}{p}\right) = \left(\frac{d}{\pi}\right)_4^2 = \left(\frac{d}{\bar{\pi}}\right)_4^2$

(ii) $\left(\frac{d}{\pi}\right)_4^3 = \left(\frac{d}{\bar{\pi}}\right)_4$

(iii) $\left(\frac{-4}{\pi}\right)_4 = \left(\frac{-4}{\bar{\pi}}\right)_4 = 1$ since $-4 = (1+i)^4$ — a fourth power.

This gives the required value of S .

Case 2: $p \equiv 5 \pmod{8}$. In this case as before

$$\begin{aligned} \sum \chi(v) \chi(v+1) &= [N - L + R + M - 2S] + i[2R - 2M] \\ &= -R(1, 1) \text{ (see [3])} \end{aligned}$$

and $\sum \chi^2(v) \chi^2(v+1) = 3N + L - 2S - R - M = -1$,
using (3.4)', (3.5)', (3.6)', and therefore

$$S = \left(\frac{-a}{p}\right) \left[-\chi(-4a) R(1, 1) - \bar{\chi}(-4a) \overline{R(1, 1)} - 1 \cdot \chi^2(-4a) \right].$$

$$\begin{aligned} \text{Here } R(1, 1) &= (L - M - R - N + 2S) + i(2M - 2R) \\ &= (-2f + 8(1, 0) + 1) + i(2M - 2R) \text{ (see [3])}. \end{aligned}$$

We put $R(1, 1) = \pi$, then $\pi \equiv 1 \pmod{(2+2i)}$ and we get

$$S = \left(\frac{-a}{p}\right) \left[-\chi(-4a) \pi - \bar{\chi}(-4a) \bar{\pi} - \chi^2(-4a) \right]$$

and as before this

$$= -\left(\frac{a}{\bar{\pi}}\right)_4 \pi - \left(\frac{a}{\pi}\right)_4 \bar{\pi} - 1,$$

as required. This completes the proof of the Theorem.