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### 3. CYCLOTOMY FOR $p = 1 + 4f$ .

Let  $g$  be a primitive root mod  $p$  which we have already fixed in § 2. Divide the non-zero residues mod  $p$  into four classes  $A_0, A_1, A_2, A_3$  by putting  $m \equiv g^v$  in  $A_i$  if  $v \equiv i \pmod{4}$ . The cyclotomic constants  $(h, k)$  ( $0 \leq h, k \leq 3$ ) are defined to be the number of values of  $y, 1 \leq y \leq p - 2$  for which

$$(3.1) \quad y \equiv g^{4t+h} \pmod{p}, \quad 1 + y \equiv g^{4s+k} \pmod{p}$$

[i.e. for which  $y \in A_h, 1 + y \in A_k$ ].

As results differ in the two cases  $p \equiv 1 \pmod{8}$  and  $p \equiv 5 \pmod{8}$  we look at these cases separately.

*Case 1:  $p \equiv 1 \pmod{8}$ .* In this case  $p = 1 + 4f$  where  $f$  is even. We know [3] that

$$(3.2) \quad \begin{cases} (h, k) = (k, h) \\ (h, k) = (-h, k-h) \end{cases}$$

Thus  $(1, 2) = (2, 3) = (1, 3); (1, 1) = (0, 3); (2, 2) = (0, 2); (3, 3) = (0, 1)$ . Therefore of the 16 cyclotomic constants which may be written as a  $(4 \times 4)$  matrix, only five are different and we have

$$(3.3) \quad \begin{bmatrix} (0, 0) & (1, 0) & (2, 0) & (3, 0) \\ (0, 1) & (1, 1) & (2, 1) & (3, 1) \\ (0, 2) & (1, 2) & (2, 2) & (3, 2) \\ (0, 3) & (1, 3) & (2, 3) & (3, 3) \end{bmatrix} = \begin{bmatrix} A & D & C & B \\ D & B & E & E \\ C & E & C & E \\ B & E & E & D \end{bmatrix}$$

Consider the numbers  $1, 2, \dots, p - 1$ . Each  $y \in A_0$  (there are  $f$  such  $y$ 's) except the last (i.e.  $p - 1$  which is in  $A_0$  in this case) is followed by  $y + 1$  which may belong to  $A_0, A_1, A_2$  or  $A_3$ .

Similarly each  $y \in A_1$  without exception is followed by  $y + 1$  which may belong to  $A_0, A_1, A_2$  or  $A_3$  and so on. Hence we get

$$(3.4) \quad A + D + C + B = f - 1$$

$$(3.5) \quad D + B + 2E = f$$

$$(3.6) \quad 2C + 2E = f.$$

Case 2:  $p \equiv 5 \pmod{8}$ . In this case  $p = 4f + 1$  where  $f$  is odd. Now look at the congruence

$$(3.1)' \quad 1 + g^{4t+h} + g^{4s+k} \equiv 0 \pmod{p}.$$

Denote the number of solutions of (3.1)' by  $\{h, k\}$ . Then clearly  $\{h, k\} = \{k, h\}$  and the following relations are known [3]

$$(3.2)' \quad \begin{cases} \{-h, k-h\} = \{h, k\} \text{ for any } f \text{ even or odd} \\ \{h, k\} = (h, k+2) \text{ for } f \text{ odd.} \end{cases}$$

Thus  $\{1, 0\} = \{3, 3\}$ ;  $\{3, 0\} = \{1, 1\}$ ;  $\{2, 0\} = \{2, 2\}$  and  $\{3, 1\} = \{1, 2\} = \{3, 2\}$ .

Therefore the matrix of the cyclotomic constants  $\{h, k\}$  can be written as

$$(3.3)' \quad \begin{bmatrix} \{0, 0\} & \{1, 0\} & \{2, 0\} & \{3, 0\} \\ \{0, 1\} & \{1, 1\} & \{2, 1\} & \{3, 1\} \\ \{0, 2\} & \{1, 2\} & \{2, 2\} & \{3, 2\} \\ \{0, 3\} & \{1, 3\} & \{2, 3\} & \{3, 3\} \end{bmatrix} = \begin{bmatrix} L & M & N & R \\ M & R & S & S \\ N & S & N & S \\ R & S & S & M \end{bmatrix}$$

Since  $f$  is odd,  $p - 1$  belongs to  $A_2$  hence in this case as before

$$\begin{aligned} (0, 1) + (0, 1) + (0, 2) + (0, 3) &= f \\ (1, 0) + (1, 1) + (1, 2) + (1, 3) &= f \\ (2, 0) + (2, 1) + (2, 2) + (2, 3) &= f - 1. \end{aligned}$$

Now using (3.2)' and (3.3)' we get

$$(3.4)' \quad L + M + N + R = f$$

$$(3.5)' \quad R + M + 2S = f$$

$$(3.6)' \quad 2N + 2S = f - 1.$$

#### 4. THE JACOBI FUNCTION

Let  $\alpha$  be any root ( $\neq 1$ ) of  $\alpha^{p-1} = 1$ . Write

$$(4.1) \quad F(\alpha) = \sum_{k=0}^{p-2} \alpha^k \zeta^{qk} \text{ where } \zeta^p = 1 \text{ and } \zeta \neq 1.$$