

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 20 (1974)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: CONSTRUCTION OF GAUSS
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Kapitel: 2. Continued Fractions
DOI: <https://doi.org/10.5169/seals-46891>

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A CONSTRUCTION OF GAUSS

by C. W. BARNES

1. INTRODUCTION

Every prime of the form $4n + 1$ can be expressed uniquely as the sum of two squares. Suppose $p = x^2 + y^2$ where p is a prime of the form $4n + 1$. A construction for x and y was given by Legendre [8] in terms of the continued fraction for \sqrt{p} . In [1] we gave a new construction for x and y , again using the continued fraction for \sqrt{p} . A summary of the various constructions is given in Davenport [5], pages 120-123.

Gauss [6] remarked that if $p = 4n + 1$, and if α and β are defined by $\beta \equiv \frac{(2n)!}{2(n!)^2} \pmod{p}$, $\alpha \equiv (2n)! \beta \pmod{p}$, where $|\alpha| < \frac{p}{2}$, $|\beta| < \frac{p}{2}$ then $p = \alpha^2 + \beta^2$; a particularly simple construction to state. Proofs of the construction of Gauss were given by Cauchy [4], page 414, and Jacobsthal [7]; however, neither of them is simple.

In the present note we give a simple proof of the construction of Gauss based on the method in [1].

2. CONTINUED FRACTIONS

We continue with the notation in [1]. The results we need can be found in Perron [9]. We denote the simple continued fraction

$$(1) \quad a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \frac{1}{a_5 + \frac{1}{a_6 + \frac{1}{a_7 + \frac{1}{a_8 + \frac{1}{a_9 + \frac{1}{a_n}}}}}}}}}}}}$$

by $[a_0, a_1, \dots, a_n]$. For $0 \leq m \leq n$ we denote the numerator and denominator of the m^{th} approximant to $[a_0, a_1, \dots, a_n]$ by A_m and B_m respectively.

If p is a prime of the form $4n + 1$, then

$$(2) \quad \sqrt{p} = [a_0, \overline{a_1, \dots, a_m, a_m, \dots, a_1}, 2a_0]$$

in the usual notation for a periodic continued fraction. The symmetric part of the period does not have a central term. In [1] we proved that $p = x^2 + y^2$ where

$$(3) \quad x = pB_m B_{m-1} - A_m A_{m-1}$$

$$(4) \quad y = A_m^2 - pB_m^2$$

and where $\frac{A_m}{B_m}$ is the m^{th} approximant to (2). We also showed that

$$(5) \quad p = \frac{A_m^2 + A_{m-1}^2}{B_m^2 + B_{m-1}^2}.$$

3. THE QUADRATIC CHARACTER OF

$$\frac{(2n)!}{2(n!)^2}.$$

It is well known that if p is a prime of the form $4n + 1$ then $\left\{ \left(\frac{p-1}{2} \right)! \right\}^2 \equiv -1 \pmod{p}$; that is, $(2n)!^2 \equiv -1 \pmod{p}$. We make use of this in the

LEMMA. If $p = 4n + 1$ is a prime then $\frac{(2n)!}{2(n!)^2}$ is a quadratic residue of p .

Proof. We use Euler's criterion. Thus if we suppose that $\frac{(2n)!}{2(n!)^2}$ is a quadratic nonresidue of p we have $\left\{ \frac{(2n)!}{2(n!)^2} \right\}^{\frac{p-1}{2}} \equiv -1 \pmod{p}$ and thus $\left\{ (2n)!^2 \right\}^{\frac{p-1}{4}} \equiv - \left\{ 2(n!)^2 \right\}^{\frac{p-1}{2}} \pmod{p}$. Since $(2n)!^2 \equiv -1 \pmod{p}$ and $n!^{p-1} \equiv 1 \pmod{p}$ we have $(-1)^n \equiv -2 \frac{p-1}{2} \pmod{p}$, or $(-1)^{n+1} \equiv (-1)^{\frac{p^2+1}{8}}$, using the standard result for the quadratic character of 2 with res-