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# A CONSTRUCTION OF GAUSS

by C. W. BARNES

## 1. INTRODUCTION

Every prime of the form  $4n + 1$  can be expressed uniquely as the sum of two squares. Suppose  $p = x^2 + y^2$  where  $p$  is a prime of the form  $4n + 1$ . A construction for  $x$  and  $y$  was given by Legendre [8] in terms of the continued fraction for  $\sqrt{p}$ . In [1] we gave a new construction for  $x$  and  $y$ , again using the continued fraction for  $\sqrt{p}$ . A summary of the various constructions is given in Davenport [5], pages 120-123.

Gauss [6] remarked that if  $p = 4n + 1$ , and if  $\alpha$  and  $\beta$  are defined by  $\beta \equiv \frac{(2n)!}{2(n!)^2} \pmod{p}$ ,  $\alpha \equiv (2n)! \beta \pmod{p}$ , where  $|\alpha| < \frac{p}{2}$ ,  $|\beta| < \frac{p}{2}$  then  $p = \alpha^2 + \beta^2$ ; a particularly simple construction to state. Proofs of the construction of Gauss were given by Cauchy [4], page 414, and Jacobsthal [7]; however, neither of them is simple.

In the present note we give a simple proof of the construction of Gauss based on the method in [1].

## 2. CONTINUED FRACTIONS

We continue with the notation in [1]. The results we need can be found in Perron [9]. We denote the simple continued fraction

$$(1) \quad a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \frac{1}{a_5 + \frac{1}{a_6 + \frac{1}{a_7 + \frac{1}{a_8 + \frac{1}{a_9 + \frac{1}{a_n}}}}}}}}}}$$

by  $[a_0, a_1, \dots, a_n]$ . For  $0 \leq m \leq n$  we denote the numerator and denominator of the  $m^{\text{th}}$  approximant to  $[a_0, a_1, \dots, a_n]$  by  $A_m$  and  $B_m$  respectively.

If  $p$  is a prime of the form  $4n + 1$ , then

$$(2) \quad \sqrt{p} = [a_0, \overline{a_1, \dots, a_m, a_m, \dots, a_1}, 2a_0]$$

in the usual notation for a periodic continued fraction. The symmetric part of the period does not have a central term. In [1] we proved that  $p = x^2 + y^2$  where

$$(3) \quad x = pB_m B_{m-1} - A_m A_{m-1}$$

$$(4) \quad y = A_m^2 - pB_m^2$$

and where  $\frac{A_m}{B_m}$  is the  $m^{\text{th}}$  approximant to (2). We also showed that

$$(5) \quad p = \frac{A_m^2 + A_{m-1}^2}{B_m^2 + B_{m-1}^2}.$$

### 3. THE QUADRATIC CHARACTER OF

$$\frac{(2n)!}{2(n!)^2}.$$

It is well known that if  $p$  is a prime of the form  $4n + 1$  then  $\left\{ \left( \frac{p-1}{2} \right)! \right\}^2 \equiv -1 \pmod{p}$ ; that is,  $(2n)!^2 \equiv -1 \pmod{p}$ . We make use of this in the

LEMMA. If  $p = 4n + 1$  is a prime then  $\frac{(2n)!}{2(n!)^2}$  is a quadratic residue of  $p$ .

Proof. We use Euler's criterion. Thus if we suppose that  $\frac{(2n)!}{2(n!)^2}$  is a quadratic nonresidue of  $p$  we have  $\left\{ \frac{(2n)!}{2(n!)^2} \right\}^{\frac{p-1}{2}} \equiv -1 \pmod{p}$  and thus  $\left\{ (2n)!^2 \right\}^{\frac{p-1}{4}} \equiv - \left\{ 2(n!)^2 \right\}^{\frac{p-1}{2}} \pmod{p}$ . Since  $(2n)!^2 \equiv -1 \pmod{p}$  and  $n!^{p-1} \equiv 1 \pmod{p}$  we have  $(-1)^n \equiv -2 \frac{p-1}{2} \pmod{p}$ , or  $(-1)^{n+1} \equiv (-1)^{\frac{p^2+1}{8}}$ , using the standard result for the quadratic character of 2 with res-