

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 20 (1974)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: LATTICE POINTS INSIDE A CONVEX BODY
Autor: Chakerian, G. D.
DOI: <https://doi.org/10.5169/seals-46908>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 08.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

LATTICE POINTS INSIDE A CONVEX BODY

by G. D. CHAKERIAN

A lattice point in the Euclidean plane \mathbf{R}^2 is a point with integer coordinates. H. Steinhaus posed as an elementary problem the proof that for each natural number n there exists a circle in the plane with exactly n lattice points in its interior. The proof of this, as given for example in [1], [3], [4], [5], uses the fact that there exists a point P in the plane such that any circle centered at P passes through at most one lattice point. Then the result is obtained by considering a continuously expanding family of circles centered at P . Indeed, using properties of irrational numbers one can show directly that the point $P = (\sqrt{2}, \sqrt{3})$ will serve.

It was shown by Browkin that the analogous result holds for a square, and by Schinzel and Kulikowski that in fact the circle may be replaced by any plane convex body. See [1], [2], [3], [4] for references to these and related results. In this note we take a new viewpoint in establishing the existence of the crucial point P and are able to generalize the result as follows.

THEOREM. *Let K be a convex body in m -dimensional Euclidean space \mathbf{R}^m , and let S be a countably infinite isolated subset of \mathbf{R}^m . Then for each natural number n there exists a homothetic copy of K with exactly n points of S in its interior.*

Before proving the theorem, let us consider another way to approach the problem of Steinhaus. For each pair of distinct lattice points A and B , let l be the perpendicular bisector of the line segment AB . Note that any circle passing through both A and B must have its center on l . Thus if there is a point P not belonging to any of the lines l , then any circle centered at P passes through at most one lattice point. But the required point P certainly exists, since there are only countably many such lines l , and the plane cannot be covered by a countable number of lines. [If one wants to crack a peanut with a sledgehammer, then observe that each line l is a nowhere dense set, and the Baire Category Theorem implies the plane is not the union of a countable number of such sets. An elementary argument follows by choosing a line m not belonging to the given set of lines. Each of the given

lines l intersects m in at most one point, giving an at most countable subset of m].

It is now apparent how the argument generalizes to show that in a Euclidean space of any dimension, for each natural number n there exists a sphere with exactly n lattice points in its interior. Simply observe that the collection of hyperplanes that are perpendicular bisectors of segments AB , where A and B are lattice points, is a countable collection of nowhere dense sets and hence cannot cover the entire space. Hence there is a point P not belonging to any of these hyperplanes. Thus any sphere centered at P passes through at most one lattice point, and the result follows.

One observes that the argument used for the existence of P required no special property of lattice points except that they comprise a countable set, and the expanding sphere centered at P contains a finite set of lattice points at any stage because the lattice points are isolated. Thus a stronger result is implied. Namely, given any countable infinite isolated subset S of a Euclidean space, there exists a point P such that for each natural number n there exists a sphere centered at P with exactly n points of S in its interior.

This brings us to the proof of the main theorem. Let K be a convex body in m -dimensional Euclidean space \mathbf{R}^m . That is, K is a compact convex subset with nonempty interior. Suppose the origin is interior to K . The gauge function $f: \mathbf{R}^m \rightarrow \mathbf{R}$ of K is defined by $f(x) = \inf \{ \mu > 0 : x/\mu \in K \}$, $x \in \mathbf{R}^m$. Note that a point x belongs to the boundary of K if and only if $f(x) = 1$. If $x \in \mathbf{R}^m$ and $\lambda > 0$, then $x + \lambda K$, the set of points of the form $x + \lambda y$ for $y \in K$, is homothetic to K . A point a belongs to the boundary of $x + \lambda K$ if and only if $f(a-x) = \lambda$. Given any two points a and b , the boundary of $x + \lambda K$ contains both a and b if and only if $f(a-x) = \lambda = f(b-x)$. The locus of points x such that the boundary of $x + \lambda K$ contains both a and b for some λ is thus equal to $C(a, b) = \{x : f(a-x) = f(b-x)\}$. But for each fixed a and b one has that $C(a, b)$ is a nowhere dense subset of \mathbf{R}^m . [Observe that the graph $\{(x, \lambda) \in \mathbf{R}^{m+1} : \lambda = f(x), x \in \mathbf{R}^m\}$ is the boundary of a convex cone in \mathbf{R}^{m+1} . The set $C(a, b)$ is the projection into \mathbf{R}^m of the intersection of a certain distinct pair of translates of this graph, and it is not difficult to show this is nowhere dense in \mathbf{R}^m .] Now let S be a countably infinite isolated subset of \mathbf{R}^m . The collection of sets $C(a, b)$, for $a, b \in S$, is countable, and hence does not cover \mathbf{R}^m since each set is nowhere dense. Thus there exists a point x_0 such that the boundary of $x_0 + \lambda K$ contains at most one point of S for each $\lambda > 0$. For a sufficiently small value of λ the body $x_0 + \lambda K$ contains exactly one lattice point. The theorem follows by choosing successively larger values of λ tending to infinity.

Finally observe that we have obtained a slightly stronger result than stated in the theorem, since all the homothetic copies of K may be chosen of the form $x_0 + \lambda K$, with x_0 fixed.

REFERENCES

- [1] HONSBERGER, R. *Mathematical Gems, Dolciani Math. Expositions, No. 1, Math. Assoc. of Amer.*, 1974.
- [2] SCHINZEL, A. Sur l'existence d'un cercle passant par un nombre donné de points aux coordonnées entières. *Enseignement Math. (2) 4* (1958), pp. 71-72.
- [3] SIERPINSKI, W. *A Selection of Problems in the Theory of Numbers*. Warsaw, 1964.
- [4] — Sur quelques problèmes concernant les points aux coordonnées entières, *Enseignement Math. (2) 4* (1959), pp. 25-31.
- [5] STEINHAUS, H. *One Hundred Problems in Elementary Mathematics*. New York, 1964.

(Reçu le 2 juillet 1974)

G. D. Chakerian

Department of Mathematics
University of California
Davis, California, 95616

Vide-leer-empty