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LATTICE POINTS INSIDE A CONVEX BODY

by G. D. Chakerian

A lattice point in the Euclidean plane $\mathbf{R}^2$ is a point with integer coordinates. H. Steinhaus posed as an elementary problem the proof that for each natural number n there exists a circle in the plane with exactly n lattice points in its interior. The proof of this, as given for example in [1], [3], [4], [5], uses the fact that there exists a point P in the plane such that any circle centered at P passes through at most one lattice point. Then the result is obtained by considering a continuously expanding family of circles centered at P. Indeed, using properties of irrational numbers one can show directly that the point $P = (\sqrt{2}, \sqrt{3})$ will serve.

It was shown by Browkin that the analogous result holds for a square, and by Schinzel and Kulikowski that in fact the circle may be replaced by any plane convex body. See [1], [2], [3], [4] for references to these and related results. In this note we take a new viewpoint in establishing the existence of the crucial point P and are able to generalize the result as follows.

THEOREM. *Let K be a convex body in m-dimensional Euclidean space $\mathbf{R}^m$, and let S be a countably infinite isolated subset of $\mathbf{R}^m$. Then for each natural number* n *there exists a homothetic copy of K with exactly* n *points of S in its interior.*

Before proving the theorem, let us consider another way to approach the problem of Steinhaus. For each pair of distinct lattice points A and B, let l be the perpendicular bisector of the line segment AB. Note that any circle passing through both A and B must have its center on l. Thus if there is a point P not belonging to any of the lines l, then any circle centered at P passes through at most one lattice point. But the required point P certainly exists, since there are only countably many such lines l, and the plane cannot be covered by a countable number of lines. [If one wants to crack a peanut with a sledgehammer, then observe that each line l is a nowhere dense set, and the Baire Category Theorem implies the plane is not the union of a countable number of such sets. An elementary argument follows by choosing a line m not belonging to the given set of lines. Each of the given lines l intersects m in at most one point, giving an at most countable subset of m].

It is now apparent how the argument generalizes to show that in a Euclidean space of any dimension, for each natural number n there exists a sphere with exactly n lattice points in its interior. Simply observe that the collection of hyperplanes that are perpendicular bisectors of segments AB, where A and B are lattice points, is a countable collection of nowhere dense sets and hence cannot cover the entire space. Hence there is a point P not belonging to any of these hyperplanes. Thus any sphere centered at P passes through at most one lattice point, and the result follows.

One observes that the argument used for the existence of P required no special property of lattice points except that they comprise a countable set, and the expanding sphere centered at P contains a finite set of lattice points at any stage because the lattice points are isolated. Thus a stronger result is implied. Namely, given any countable infinite isolated subset S of a Euclidean space, there exists a point P such that for each natural number n there exists a sphere centered at P with exactly n points of S in its interior.

This brings us to the proof of the main theorem. Let K be a convex body in m-dimensional Euclidean space $\mathbf{R}^m$. That is, K is a compact convex subset with nonempty interior. Suppose the origin is interior to K. The gauge function $f : \mathbf{R}^m \to \mathbf{R}$ of K is defined by $f(x) = \inf\{\mu > 0 : x/\mu \in K\}$, $x \in \mathbf{R}^m$. Note that a point x belongs to the boundary of K if and only if $f(x) = 1$. If $x \in \mathbf{R}^m$ and $\lambda > 0$, then $x + \lambda K$, the set of points of the form $x + \lambda y$ for $y \in K$, is homothetic to K. A point a belongs to the boundary of $x + \lambda K$ if and only if $f(a-x) = \lambda$. Given any two points a and b, the boundary of $x + \lambda K$ contains both a and b if and only if $f(a-x) = \lambda = f(b-x)$. The locus of points x such that the boundary of $x + \lambda K$ contains both a and b for some λ is thus equal to $C(a, b) = \{x : f(a-x) = f(b-x)\}$. But for each fixed a and b one has that $C(a, b)$ is a nowhere dense subset of $\mathbf{R}^m$. [Observe that the graph $\{(x, \lambda) \in \mathbf{R}^{m+1} : \lambda = f(x), x \in \mathbf{R}^m\}$ is the boundary of a convex cone in $\mathbf{R}^{m+1}$. The set $C(a, b)$ is the projection into $\mathbf{R}^m$ of the intersection of a certain distinct pair of translates of this graph, and it is not difficult to show this is nowhere dense in $\mathbf{R}^m$.] Now let S be a countably infinite isolated subset of $\mathbf{R}^m$. The collection of sets $C(a, b)$, for $a, b \in S$, is countable, and hence does not cover $\mathbf{R}^m$ since each set is nowhere dense. Thus there exists a point x_0 such that the boundary of $x_0 + \lambda K$ contains at most one point of S for each $\lambda > 0$. For a sufficiently small value of λ the body $x_0 + \lambda K$ contains exactly one lattice point. The theorem follows by choosing successively larger values of λ tending to infinity.

Finally observe that we have obtained a slightly stronger result than stated in the theorem, since all the homothetic copies of K may be chosen of the form $x_o + \lambda K$, with x_o fixed.

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