

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 19 (1973)  
**Heft:** 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** AN ELEMENTARY PROOF THAT ELLIPTIC CURVES ARE ABELIAN VARIETIES  
**Autor:** Olson, Loren D.  
**Kapitel:** § 2. Algebraic and geometric group laws on an elliptic curve  
**DOI:** <https://doi.org/10.5169/seals-46291>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 19.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

A proof of Theorem 1 may be found on page 28 of Serre [4].

Recall that:

- (1)  $\deg K = 2g - 2$  where  $K$  denotes the canonical divisor on  $X$ ,
- (2) the Riemann-Roch theorem, i.e.  $l(D) = \deg D + 1 - g + l(K - D)$  where  $l(D) = \dim_k L(D)$ , and
- (3) if  $X$  is a non-singular plane curve of degree  $n$ , then  $g = (n-1)(n-2)/2$ .

*Def.*  $X$  is an *elliptic curve* if  $g = 1$ .

Notice that if  $D$  is a divisor of degree  $n$  on a curve  $X$ , then  $n < 0 \Rightarrow L(D) = 0 \Rightarrow l(D) = 0$ . In particular, on an elliptic curve  $X$ ,  $n > 0 \Rightarrow \deg(K - D) = -n < 0 \Rightarrow l(K - D) = 0 \Rightarrow l(D) = n$  from (1) and (2) above.

*Theorem 2* A non-singular complete curve  $C$  in  $\mathbf{P}^2$  of degree 3 is an elliptic curve.

*Proof:*

$$(3) \Rightarrow g = (3-1)(3-2)/2 = 1.$$

*Theorem 3* Every elliptic curve  $X$  is isomorphic to a non-singular complete irreducible curve  $C$  in  $\mathbf{P}^2$  of degree 3.

*Proof:*

Let  $D$  be a divisor of degree 3 on  $X$ .

Theorem 1 implies that  $D$  is very ample, i.e. that we have an isomorphism from  $X$  to a non-singular complete irreducible curve  $C$  in  $\mathbf{P}(L(D))$ . Riemann-Roch  $\Rightarrow l(D) = 3 \Rightarrow \mathbf{P}(L(D)) = \mathbf{P}^2$ . Let  $n = g(C)$ .  $X$  an elliptic curve  $\Rightarrow 1 = g(X) = g(C) = (n-1)(n-2)/2 \Rightarrow n = 3$ .

Thus we have established the desired connection between (I) and (II).

## § 2. ALGEBRAIC AND GEOMETRIC GROUP LAWS ON AN ELLIPTIC CURVE

Let  $X$  be an elliptic curve over  $k$ , and let  $X(k)$  denote the set of  $k$ -points of  $X$ . We begin by defining a group law on  $X(k)$  in a rather algebraic fashion. Let  $\text{Div}^0(X)$  be the group of divisors of degree 0 on  $X$ . Let  $\sim$  denote linear equivalence, and let  $\text{Div}^0(X)/\sim$  be the quotient group. If  $D \in \text{Div}^0(X)$ , let  $\text{Cl}(D)$  be its image in  $\text{Div}^0(X)/\sim$ .

Recall that a divisor  $D = \sum n_P P$  is called effective if  $n_P \geq 0$  for all  $P$ .

*Lemma 4* Let  $D_1$  and  $D_2$  be effective divisors of degree 1 on  $X$ . Then

$$(4) D_1 = D_2 \Leftrightarrow D_1 \sim D_2.$$

*Proof:*

( $\Rightarrow$ ) Obvious.

( $\Leftarrow$ )  $D_1$  effective  $\Rightarrow L(D_1)$  contains all the constant functions.  $\deg(D_1) = 1 \Rightarrow l(D_1) = 1 \Rightarrow L(D_1)$  consists solely of the constant functions. Suppose now that  $D_1 \sim D_2$ . Then there exists  $f \in k(X)$  such that  $D_1 + (f) = D_2$ .  $D_2$  effective  $\Rightarrow f \in L(D_1) \Rightarrow f$  constant  $\Rightarrow D_1 = D_2$ .

Fix a  $k$ -point  $e$  of  $X$ . Define a map  $\Phi$  from  $X(k)$  to  $\text{Div}^0(X)/\sim$  by  $P \rightarrow \text{Cl}(P-e)$ .

*Proposition 5* The map  $\Phi : X(k) \rightarrow \text{Div}^0(X)/\sim$  is a bijection.

*Proof:*

Claim  $\Phi$  is injective. Let  $P_1, P_2 \in X(k)$ .  $\Phi(P_1) = \Phi(P_2) \Leftrightarrow \text{Cl}(P_1-e) = \text{Cl}(P_2-e) \Leftrightarrow P_1 - e \sim P_2 - e \Leftrightarrow P_1 \sim P_2 \Leftrightarrow P_1 = P_2$ . So  $\Phi$  is injective. Claim  $\Phi$  is surjective. Let  $\bar{D} \in \text{Div}^0(X)/\sim$  with  $D \in \text{Div}^0(X)$  such that  $\text{Cl}(D) = \bar{D}$ .  $\deg(D+e) = 1 \Rightarrow l(D+e) = 1 \Rightarrow$  there exists  $f \in L(D+e)$ ,  $f \neq 0$ , such that  $(f) + D + e \geq 0$ , i.e.  $(f) + D + e = P$  for  $P \in X(k)$ .  $\Phi(P) = \text{Cl}(P-e) = \text{Cl}((f)+D) = \text{Cl}(D) = \bar{D}$ . Therefore  $\Phi$  is surjective, and hence bijective.

Thus  $X(k)$  receives an abelian group structure via  $\Phi$ , i.e. the sum of  $P_1$  and  $P_2$  is  $\Phi^{-1}(\Phi(P_1) + \Phi(P_2)) = \Phi^{-1}(\text{Cl}(P_1-e) + \text{Cl}(P_2-e)) = \Phi^{-1}(\text{Cl}(P_1+P_2-2e)) =$  that point  $Q$  on  $X$  such that  $Q \sim P_1 + P_2 - e$ . We therefore have a map  $M : X(k) \times X(k) \rightarrow X(k)$  which we shall call the “algebraic” group law.

Now let us assume that  $C$  is a non-singular complete cubic in  $\mathbf{P}^2$ . We proceed to define a “geometric” group law on  $C(k)$ . If  $P_1, P_2 \in C(k)$ , there exists a unique line  $L$  such that the intersection cycle  $L \cdot C = P_1 + P_2 + P_3$  for some  $P_3 \in C(k)$ . If  $P_1 \neq P_2$ ,  $L$  is the unique line through  $P_1$  and  $P_2$ . If  $P_1 = P_2$ ,  $L$  is the unique tangent to  $C$  at  $P_1$ .  $P_3$  is thus uniquely determined by  $P_1$  and  $P_2$  and we have defined a mapping  $\varphi : C(k) \times C(k) \rightarrow C(k)$ . Let  $e$  be a fixed  $k$ -point of  $C$ . By repeating the preceding procedure with the points  $\varphi(P_1, P_2)$  and  $e$ , we will obtain a new point  $P_1 + P_2$ . Let  $m : C(k) \times C(k) \rightarrow C(k)$  be the resulting map, i.e.  $m$  is the composition of  $(e, \varphi)$  and  $\varphi$ ,  $m = \varphi^\circ(e, \varphi)$ .  $m$  is the “geometric” group law.

By using certain geometric properties of  $\mathbf{P}^2$ , it is possible to prove that  $m$  gives  $C(k)$  an abelian group structure (cf. Fulton [1], p. 125). We choose instead to prove the following proposition.

*Proposition 6* The “algebraic” group law on  $C$  coincides with the “geometric” group law on  $C$ , i.e.  $m = M$ .

*Proof:*

Let  $P_1, P_2 \in C(k)$ . Let  $P_3 = \varphi(P_1, P_2)$ . Then there exists a line  $L_1$  such that  $L_1 \cdot C = P_1 + P_2 + P_3$ . Let  $P_4 = \varphi(e, P_3) = \varphi(e, \varphi(P_1, P_2)) = m(P_1, P_2)$ . Then there exists a line  $L_2$  such that  $L_2 \cdot C = e + P_3 + P_4$ . Let  $f = L_1/L_2$  and regard  $f$  as an element of  $k(C)$ .  $(f) = P_1 + P_2 - e - P_4 \Rightarrow P_4 \sim P_1 + P_2 - e$ , i.e.  $P_4 = M(P_1, P_2)$ . Therefore  $m = M$ .

### § 3. ELLIPTIC CURVES AND ABELIAN VARIETIES

The purpose of this section is to prove the equivalence of notions (II) and (III). Up to this point, we have a group law on the set of  $k$ -points of an elliptic curve, and we would like to know that this is induced by an abelian variety structure. We shall also prove that 1-dimensional abelian varieties are elliptic curves.

*Definition* Let  $k$  be a field. An *abelian variety*  $X$  is a complete non-singular variety defined over  $k$  together with  $k$ -morphisms

$$\begin{aligned} m : X \times X &\rightarrow X \\ i : X &\rightarrow X \\ e : \operatorname{Spec}(k) &\rightarrow X \end{aligned}$$

which satisfy the usual group axioms (cf. Mumford [2], p. 95).

To show that an elliptic curve can be given the structure of an abelian variety, it suffices to check that the map  $\varphi$  described in § 2 is a morphism. Recall that  $\varphi$  was defined on  $k$ -points as taking  $(P_1, P_2) \in C(k) \times C(k)$  to the unique third point  $P_3 \in C(k)$  such that  $P_1 + P_2 + P_3 = L \cdot C$  for some line  $L$ . It is quite easy to see that  $\varphi$  is a morphism on a certain affine open subset of  $C \times C$ . To be precise, we have the following lemma.

*Lemma 7*  $\varphi$  defines a morphism from

$$\mathcal{S} = \operatorname{Spec}(k[X_1, Y_1, X_2, Y_2]/(f(X_1, Y_1), f(X_2, Y_2))(X_1 - X_2))$$

to  $\mathcal{T} = \operatorname{Spec}(k[X_3, Y_3]/f(X_3, Y_3))$  (where  $f$  is an affine equation for  $C$ ).