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$$(2.8) \quad \Psi(\chi_E, x) \rightarrow 0$$

for every bounded measurable subset  $E$  of  $R^+$ , and

$$(2.9) \quad W_\Psi(1, x) = O(1).$$

### 3. TRANSFORMATIONS OF $O$ -REGULAR AND SLOWLY VARYING FUNCTIONS BY REGULAR OPERATORS.

3.1. The class of positive functions which are eventually bounded away from zero and infinity has been extended to the class of  $O$ -regular functions defined as follows:

A positive, measurable function  $l$  on  $R^+$  is  $O$ -regular if

$$(3.1) \quad \frac{l(\lambda x)}{l(x)} = O(1) \quad (x \rightarrow \infty)$$

for every  $\lambda > 0$ .

For example, any function  $l$  such that  $ax^\alpha \leq l(x) \leq Ax^\alpha$ , where  $\alpha \in R$ , clearly satisfies condition (3.1).

The class of  $O$ -regular functions and related classes of functions have been studied extensively by V. G. Avakumović [8, 9, 10, 11], J. Karamata [14], N. K. Bari, S. B. Stečkin [15], M. A. Krasnoselskiĭ, T. B. Rutickiĭ [16], W. Matuszewska [17] and others.

The closely related class of slowly varying ( $SV$ ) functions, introduced by J. Karamata ([12], [13]), generalizes the class of functions converging to a positive limit. A positive, measurable function  $L$  defined on  $R^+$  is a slowly varying function if

$$(3.2) \quad \lim_{x \rightarrow \infty} \frac{L(\lambda x)}{L(x)} = 1$$

for every  $\lambda > 0$ .

Clearly, every measurable function on  $R^+$  which converges to a positive limit as  $x \rightarrow \infty$  is a  $SV$  function. Also, functions like

$$\varphi(x) = \begin{cases} 1, & 0 \leq x < e, \\ \log x, & x \geq e, \end{cases}, \quad h(x) = \left(2 + \frac{\sin x}{x}\right) \varphi(x),$$

and their iterations are  $SV$  functions. More generally, any measurable function  $g$  on  $R^+$  such that  $\varphi(x) \leq g(x) \leq \varphi(x) + \sqrt{\varphi(x)}$  is a  $SV$  function.

The most important properties of  $O$ -regular and  $SV$  functions can be stated as follows:

REPRESENTATION THEOREMS: *If  $l$  is an  $O$ -regular function, there exist  $B > 0$  and bounded measurable functions  $\alpha$  and  $\beta$  on  $[B, \infty]$  such that*

$$(3.3) \quad l(x) = \exp \left( \alpha(x) + \int_B^x \frac{\beta(t)}{t} dt \right) \text{ for } x \geq B.$$

*If  $L$  is a  $SV$  function, then for some  $B > 0$ ,*

$$(3.4) \quad L(x) = \exp \left( \eta(x) + \int_B^x \frac{\varepsilon(t)}{t} dt \right) \text{ for } x \geq B,$$

*where  $\eta$  and  $\varepsilon$  are bounded measurable functions on  $[B, \infty]$  such that  $\eta(x) \rightarrow c$  and  $\varepsilon(x) \rightarrow 0$  ( $x \rightarrow \infty$ ).*

A proof of these results for continuous  $O$ -regular and  $SV$  functions can be found in [12], [13], and [14]. These results were subsequently extended to measurable  $O$ -regular and  $SV$  functions by a number of authors (see [18] for details).

One of the typical and simplest results about the asymptotic behavior of special linear transforms of  $SV$  functions is probably the following result of K. Knopp [19]:

If  $L$  is a  $SV$  function, and if  $L \in \mathcal{M}_0$ , then

$$\frac{1}{xL(x)} \int_0^\infty e^{-(t/x)} L(t) dt \rightarrow 1 \quad (x \rightarrow \infty).$$

Similar results involving more or less special transformations have been obtained by G. H. Hardy and W. W. Rogosinski [4], S. Aljančić, R. Bojanić, M. Tomić [20], R. Bojanić and J. Karamata [21], and, in slightly different form, by D. Drasin ([22], Th. 6). The most general result of this type, obtained by M. Vuilleumier [23], [24], can be stated as follows:

*Let  $G$  be defined by (1.1). In order that*

$$\frac{G(L, x)}{L(x)} \rightarrow 1 \quad (x \rightarrow \infty)$$

*holds for every  $SV$  function  $L \in \mathcal{M}_0$  it is necessary and sufficient that, as  $x \rightarrow \infty$ ,*

$$(i) \quad \int_0^\infty \Psi(x, t) dt \rightarrow 1,$$

(ii) *there exists  $\eta > 0$  such that*

$$\int_0^x |\Psi(x, t)| t^{-\eta} dt = O(x^{-\eta}) \text{ and } \int_x^\infty |\Psi(x, t)| t^\eta dt = O(x^\eta).$$

3.2. Theorem 1 characterizes boundedness preserving operators. A natural extension of that result is the theorem which characterizes regular operators  $\Psi$  with the property that  $\Psi(l, x) = O(l(x))$  ( $x \rightarrow \infty$ ) holds for every  $O$ -regular function  $l \in \mathcal{M}_0$ . In this direction we have the following result:

THEOREM 4. *Let  $\Psi: \mathcal{M}_0 \rightarrow \mathcal{F}_0$  be a regular operator. In order that*

$$(3.5) \quad \Psi(l, x) = O(l(x)) \quad (x \rightarrow \infty),$$

*holds for every  $O$ -regular function  $l \in \mathcal{M}_0$  it is necessary and sufficient that for all  $\alpha > 0$ , as  $x \rightarrow \infty$ ,*

$$(3.6) \quad V_\Psi(t^\alpha, x) = O(x^\alpha)$$

*and*

$$(3.7) \quad V_\Psi(\chi_{[0,1]}(t) + t^{-\alpha} \chi_{(1,\infty)}(t), x) = O(x^{-\alpha})$$

*where  $V_\Psi$  is defined by (1.5).*

Likewise, as an analog of Theorem 2, the following theorem characterizes regular operators which have the property that

$$\Psi(L, x) = O(L(x)) \quad (x \rightarrow \infty)$$

holds for every  $SV$  function  $L \in \mathcal{M}_0$ :

THEOREM 5. *Let  $\Psi: \mathcal{M}_0 \rightarrow \mathcal{F}_0$  be a regular operator. In order that*

$$(3.8) \quad \Psi(L, x) = O(L(x)) \quad (x \rightarrow \infty)$$

*holds for every  $SV$  function  $L \in \mathcal{M}_0$  it is necessary and sufficient that there exists  $\eta > 0$  such that, as  $x \rightarrow \infty$ ,*

$$(3.9) \quad W_\Psi(t^\eta, x) = O(x^\eta)$$

*and*

$$(3.10) \quad W_\Psi(\chi_{[0,1]}(t) + t^{-\eta} \chi_{(1,\infty)}(t), x) = O(x^{-\eta})$$

*where  $W_\Psi$  is defined by (2.5).*

Finally, the analog of Theorem 3 can be stated as follows:

THEOREM 6. Let  $\Psi: \mathcal{M}_0 \rightarrow \mathcal{F}_0$  be a regular operator. In order that

$$(3.11) \quad \frac{\Psi(L, x)}{L(x)} \rightarrow 1 \quad (x \rightarrow \infty)$$

holds for every SV function  $L \in \mathcal{M}_0$  it is necessary and sufficient that

$$(3.12) \quad \Psi(1, x) \rightarrow 1 \quad (x \rightarrow \infty),$$

and that the asymptotic relations (3.9) and (3.10) hold for some  $\eta > 0$ .

#### 4. PROOFS.

4.1. Proof of Theorem 1. The sufficiency of condition (2.2) follows from the inequality

$$|\Psi(f, x)| \leq V_\Psi(1, x) \|f\|.$$

The necessity of (2.2) is proved by way of contradiction. Suppose that (2.2) is not satisfied. Then

$$(4.1.1) \quad \limsup_{x \rightarrow \infty} V_\Psi(1, x) = \infty.$$

In view of (4.1.1), (2.1) and the properties of  $\Psi$ , it is possible to find by induction an increasing sequence  $(x_k)$  going to infinity and a sequence  $(g_k)$  of functions in  $\mathcal{M}_0$  such that, if  $A_k$  is defined by  $A_k = V_\Psi(1, x_k)$ , then

$$(4.1.2) \quad A_1 \geq 16 \text{ and } A_k \geq 16 A_{k-1}, \quad k = 2, 3, \dots,$$

$$(4.1.3) \quad A_k \geq 16 \left( \sup_{x \in R^+} |\Psi \left( \sum_{i=1}^{k-1} \frac{g_i}{\sqrt{A_i}}, x \right)| \right)^2, \quad k = 2, 3, \dots,$$

and

$$(4.1.4) \quad |g_k| \leq 1, \quad |\Psi(g_k, x_k)| \geq \frac{3}{4} A_k, \quad k = 1, 2, \dots$$

Let

$$(4.1.5) \quad g(x) = \sum_{i=1}^{\infty} \frac{g_i(x)}{\sqrt{A_i}}.$$