

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 19 (1973)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ASYMPTOTIC PROPERTIES OF LINEAR OPERATORS
Autor: Bojanic, R. / Vuilleumier, M.
Kapitel: 3. Transformations of 0-regular and slowly varying functions by regular operators.
DOI: <https://doi.org/10.5169/seals-46293>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 18.02.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

$$(2.8) \quad \Psi(\chi_E, x) \rightarrow 0$$

for every bounded measurable subset E of R^+ , and

$$(2.9) \quad W_\Psi(1, x) = O(1).$$

3. TRANSFORMATIONS OF O -REGULAR AND SLOWLY VARYING FUNCTIONS BY REGULAR OPERATORS.

3.1. The class of positive functions which are eventually bounded away from zero and infinity has been extended to the class of O -regular functions defined as follows:

A positive, measurable function l on R^+ is O -regular if

$$(3.1) \quad \frac{l(\lambda x)}{l(x)} = O(1) \quad (x \rightarrow \infty)$$

for every $\lambda > 0$.

For example, any function l such that $ax^\alpha \leq l(x) \leq Ax^\alpha$, where $\alpha \in R$, clearly satisfies condition (3.1).

The class of O -regular functions and related classes of functions have been studied extensively by V. G. Avakumović [8, 9, 10, 11], J. Karamata [14], N. K. Bari, S. B. Stečkin [15], M. A. Krasnoselskiĭ, T. B. Rutickiĭ [16], W. Matuszewska [17] and others.

The closely related class of slowly varying (SV) functions, introduced by J. Karamata ([12], [13]), generalizes the class of functions converging to a positive limit. A positive, measurable function L defined on R^+ is a slowly varying function if

$$(3.2) \quad \lim_{x \rightarrow \infty} \frac{L(\lambda x)}{L(x)} = 1$$

for every $\lambda > 0$.

Clearly, every measurable function on R^+ which converges to a positive limit as $x \rightarrow \infty$ is a SV function. Also, functions like

$$\varphi(x) = \begin{cases} 1, & 0 \leq x < e, \\ \log x, & x \geq e, \end{cases}, \quad h(x) = \left(2 + \frac{\sin x}{x}\right) \varphi(x),$$

and their iterations are SV functions. More generally, any measurable function g on R^+ such that $\varphi(x) \leq g(x) \leq \varphi(x) + \sqrt{\varphi(x)}$ is a SV function.

The most important properties of O -regular and SV functions can be stated as follows:

REPRESENTATION THEOREMS: *If l is an O -regular function, there exist $B > 0$ and bounded measurable functions α and β on $[B, \infty]$ such that*

$$(3.3) \quad l(x) = \exp \left(\alpha(x) + \int_B^x \frac{\beta(t)}{t} dt \right) \text{ for } x \geq B.$$

If L is a SV function, then for some $B > 0$,

$$(3.4) \quad L(x) = \exp \left(\eta(x) + \int_B^x \frac{\varepsilon(t)}{t} dt \right) \text{ for } x \geq B,$$

where η and ε are bounded measurable functions on $[B, \infty]$ such that $\eta(x) \rightarrow c$ and $\varepsilon(x) \rightarrow 0$ ($x \rightarrow \infty$).

A proof of these results for continuous O -regular and SV functions can be found in [12], [13], and [14]. These results were subsequently extended to measurable O -regular and SV functions by a number of authors (see [18] for details).

One of the typical and simplest results about the asymptotic behavior of special linear transforms of SV functions is probably the following result of K. Knopp [19]:

If L is a SV function, and if $L \in \mathcal{M}_0$, then

$$\frac{1}{xL(x)} \int_0^\infty e^{-(t/x)} L(t) dt \rightarrow 1 \quad (x \rightarrow \infty).$$

Similar results involving more or less special transformations have been obtained by G. H. Hardy and W. W. Rogosinski [4], S. Aljančić, R. Bojanić, M. Tomić [20], R. Bojanić and J. Karamata [21], and, in slightly different form, by D. Drasin ([22], Th. 6). The most general result of this type, obtained by M. Vuilleumier [23], [24], can be stated as follows:

Let G be defined by (1.1). In order that

$$\frac{G(L, x)}{L(x)} \rightarrow 1 \quad (x \rightarrow \infty)$$

holds for every SV function $L \in \mathcal{M}_0$ it is necessary and sufficient that, as $x \rightarrow \infty$,

$$(i) \quad \int_0^\infty \Psi(x, t) dt \rightarrow 1,$$

(ii) *there exists $\eta > 0$ such that*

$$\int_0^x |\Psi(x, t)| t^{-\eta} dt = O(x^{-\eta}) \text{ and } \int_x^\infty |\Psi(x, t)| t^\eta dt = O(x^\eta).$$

3.2. Theorem 1 characterizes boundedness preserving operators. A natural extension of that result is the theorem which characterizes regular operators Ψ with the property that $\Psi(l, x) = O(l(x))$ ($x \rightarrow \infty$) holds for every O -regular function $l \in \mathcal{M}_0$. In this direction we have the following result:

THEOREM 4. *Let $\Psi: \mathcal{M}_0 \rightarrow \mathcal{F}_0$ be a regular operator. In order that*

$$(3.5) \quad \Psi(l, x) = O(l(x)) \quad (x \rightarrow \infty),$$

holds for every O -regular function $l \in \mathcal{M}_0$ it is necessary and sufficient that for all $\alpha > 0$, as $x \rightarrow \infty$,

$$(3.6) \quad V_\Psi(t^\alpha, x) = O(x^\alpha)$$

and

$$(3.7) \quad V_\Psi(\chi_{[0,1]}(t) + t^{-\alpha} \chi_{(1,\infty)}(t), x) = O(x^{-\alpha})$$

where V_Ψ is defined by (1.5).

Likewise, as an analog of Theorem 2, the following theorem characterizes regular operators which have the property that

$$\Psi(L, x) = O(L(x)) \quad (x \rightarrow \infty)$$

holds for every SV function $L \in \mathcal{M}_0$:

THEOREM 5. *Let $\Psi: \mathcal{M}_0 \rightarrow \mathcal{F}_0$ be a regular operator. In order that*

$$(3.8) \quad \Psi(L, x) = O(L(x)) \quad (x \rightarrow \infty)$$

holds for every SV function $L \in \mathcal{M}_0$ it is necessary and sufficient that there exists $\eta > 0$ such that, as $x \rightarrow \infty$,

$$(3.9) \quad W_\Psi(t^\eta, x) = O(x^\eta)$$

and

$$(3.10) \quad W_\Psi(\chi_{[0,1]}(t) + t^{-\eta} \chi_{(1,\infty)}(t), x) = O(x^{-\eta})$$

where W_Ψ is defined by (2.5).

Finally, the analog of Theorem 3 can be stated as follows:

THEOREM 6. Let $\Psi: \mathcal{M}_0 \rightarrow \mathcal{F}_0$ be a regular operator. In order that

$$(3.11) \quad \frac{\Psi(L, x)}{L(x)} \rightarrow 1 \quad (x \rightarrow \infty)$$

holds for every SV function $L \in \mathcal{M}_0$ it is necessary and sufficient that

$$(3.12) \quad \Psi(1, x) \rightarrow 1 \quad (x \rightarrow \infty),$$

and that the asymptotic relations (3.9) and (3.10) hold for some $\eta > 0$.

4. PROOFS.

4.1. *Proof of Theorem 1.* The sufficiency of condition (2.2) follows from the inequality

$$|\Psi(f, x)| \leq V_\Psi(1, x) \|f\|.$$

The necessity of (2.2) is proved by way of contradiction. Suppose that (2.2) is not satisfied. Then

$$(4.1.1) \quad \limsup_{x \rightarrow \infty} V_\Psi(1, x) = \infty.$$

In view of (4.1.1), (2.1) and the properties of Ψ , it is possible to find by induction an increasing sequence (x_k) going to infinity and a sequence (g_k) of functions in \mathcal{M}_0 such that, if A_k is defined by $A_k = V_\Psi(1, x_k)$, then

$$(4.1.2) \quad A_1 \geq 16 \text{ and } A_k \geq 16 A_{k-1}, \quad k = 2, 3, \dots,$$

$$(4.1.3) \quad A_k \geq 16 \left(\sup_{x \in R^+} |\Psi(\sum_{i=1}^{k-1} \frac{g_i}{\sqrt{A_i}}, x)| \right)^2, \quad k = 2, 3, \dots,$$

and

$$(4.1.4) \quad |g_k| \leq 1, \quad |\Psi(g_k, x_k)| \geq \frac{3}{4} A_k, \quad k = 1, 2, \dots$$

Let

$$(4.1.5) \quad g(x) = \sum_{i=1}^{\infty} \frac{g_i(x)}{\sqrt{A_i}}.$$