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$$(2.8) \quad \Psi(\chi_E, x) \rightarrow 0$$

for every bounded measurable subset E of R^+ , and

$$(2.9) \quad W_\Psi(1, x) = O(1).$$

3. TRANSFORMATIONS OF O -REGULAR AND SLOWLY VARYING FUNCTIONS BY REGULAR OPERATORS.

3.1. The class of positive functions which are eventually bounded away from zero and infinity has been extended to the class of O -regular functions defined as follows:

A positive, measurable function l on R^+ is O -regular if

$$(3.1) \quad \frac{l(\lambda x)}{l(x)} = O(1) \quad (x \rightarrow \infty)$$

for every $\lambda > 0$.

For example, any function l such that $ax^\alpha \leq l(x) \leq Ax^\alpha$, where $\alpha \in R$, clearly satisfies condition (3.1).

The class of O -regular functions and related classes of functions have been studied extensively by V. G. Avakumović [8, 9, 10, 11], J. Karamata [14], N. K. Bari, S. B. Stečkin [15], M. A. Krasnoselskiĭ, T. B. Rutickiĭ [16], W. Matuszewska [17] and others.

The closely related class of slowly varying (SV) functions, introduced by J. Karamata ([12], [13]), generalizes the class of functions converging to a positive limit. A positive, measurable function L defined on R^+ is a slowly varying function if

$$(3.2) \quad \lim_{x \rightarrow \infty} \frac{L(\lambda x)}{L(x)} = 1$$

for every $\lambda > 0$.

Clearly, every measurable function on R^+ which converges to a positive limit as $x \rightarrow \infty$ is a SV function. Also, functions like

$$\varphi(x) = \begin{cases} 1, & 0 \leq x < e, \\ \log x, & x \geq e, \end{cases}, \quad h(x) = \left(2 + \frac{\sin x}{x}\right) \varphi(x),$$

and their iterations are SV functions. More generally, any measurable function g on R^+ such that $\varphi(x) \leq g(x) \leq \varphi(x) + \sqrt{\varphi(x)}$ is a SV function.

The most important properties of O -regular and SV functions can be stated as follows:

REPRESENTATION THEOREMS: *If l is an O -regular function, there exist $B > 0$ and bounded measurable functions α and β on $[B, \infty]$ such that*

$$(3.3) \quad l(x) = \exp \left(\alpha(x) + \int_B^x \frac{\beta(t)}{t} dt \right) \text{ for } x \geq B.$$

If L is a SV function, then for some $B > 0$,

$$(3.4) \quad L(x) = \exp \left(\eta(x) + \int_B^x \frac{\varepsilon(t)}{t} dt \right) \text{ for } x \geq B,$$

where η and ε are bounded measurable functions on $[B, \infty]$ such that $\eta(x) \rightarrow c$ and $\varepsilon(x) \rightarrow 0$ ($x \rightarrow \infty$).

A proof of these results for continuous O -regular and SV functions can be found in [12], [13], and [14]. These results were subsequently extended to measurable O -regular and SV functions by a number of authors (see [18] for details).

One of the typical and simplest results about the asymptotic behavior of special linear transforms of SV functions is probably the following result of K. Knopp [19]:

If L is a SV function, and if $L \in \mathcal{M}_0$, then

$$\frac{1}{xL(x)} \int_0^\infty e^{-(t/x)} L(t) dt \rightarrow 1 \quad (x \rightarrow \infty).$$

Similar results involving more or less special transformations have been obtained by G. H. Hardy and W. W. Rogosinski [4], S. Aljančić, R. Bojanić, M. Tomić [20], R. Bojanić and J. Karamata [21], and, in slightly different form, by D. Drasin ([22], Th. 6). The most general result of this type, obtained by M. Vuilleumier [23], [24], can be stated as follows:

Let G be defined by (1.1). In order that

$$\frac{G(L, x)}{L(x)} \rightarrow 1 \quad (x \rightarrow \infty)$$

holds for every SV function $L \in \mathcal{M}_0$ it is necessary and sufficient that, as $x \rightarrow \infty$,

$$(i) \quad \int_0^\infty \Psi(x, t) dt \rightarrow 1,$$

(ii) *there exists $\eta > 0$ such that*

$$\int_0^x |\Psi(x, t)| t^{-\eta} dt = O(x^{-\eta}) \text{ and } \int_x^\infty |\Psi(x, t)| t^\eta dt = O(x^\eta).$$

3.2. Theorem 1 characterizes boundedness preserving operators. A natural extension of that result is the theorem which characterizes regular operators Ψ with the property that $\Psi(l, x) = O(l(x))$ ($x \rightarrow \infty$) holds for every O -regular function $l \in \mathcal{M}_0$. In this direction we have the following result:

THEOREM 4. *Let $\Psi: \mathcal{M}_0 \rightarrow \mathcal{F}_0$ be a regular operator. In order that*

$$(3.5) \quad \Psi(l, x) = O(l(x)) \quad (x \rightarrow \infty),$$

holds for every O -regular function $l \in \mathcal{M}_0$ it is necessary and sufficient that for all $\alpha > 0$, as $x \rightarrow \infty$,

$$(3.6) \quad V_\Psi(t^\alpha, x) = O(x^\alpha)$$

and

$$(3.7) \quad V_\Psi(\chi_{[0,1]}(t) + t^{-\alpha} \chi_{(1,\infty)}(t), x) = O(x^{-\alpha})$$

where V_Ψ is defined by (1.5).

Likewise, as an analog of Theorem 2, the following theorem characterizes regular operators which have the property that

$$\Psi(L, x) = O(L(x)) \quad (x \rightarrow \infty)$$

holds for every SV function $L \in \mathcal{M}_0$:

THEOREM 5. *Let $\Psi: \mathcal{M}_0 \rightarrow \mathcal{F}_0$ be a regular operator. In order that*

$$(3.8) \quad \Psi(L, x) = O(L(x)) \quad (x \rightarrow \infty)$$

holds for every SV function $L \in \mathcal{M}_0$ it is necessary and sufficient that there exists $\eta > 0$ such that, as $x \rightarrow \infty$,

$$(3.9) \quad W_\Psi(t^\eta, x) = O(x^\eta)$$

and

$$(3.10) \quad W_\Psi(\chi_{[0,1]}(t) + t^{-\eta} \chi_{(1,\infty)}(t), x) = O(x^{-\eta})$$

where W_Ψ is defined by (2.5).

Finally, the analog of Theorem 3 can be stated as follows:

THEOREM 6. Let $\Psi: \mathcal{M}_0 \rightarrow \mathcal{F}_0$ be a regular operator. In order that

$$(3.11) \quad \frac{\Psi(L, x)}{L(x)} \rightarrow 1 \quad (x \rightarrow \infty)$$

holds for every SV function $L \in \mathcal{M}_0$ it is necessary and sufficient that

$$(3.12) \quad \Psi(1, x) \rightarrow 1 \quad (x \rightarrow \infty),$$

and that the asymptotic relations (3.9) and (3.10) hold for some $\eta > 0$.

4. PROOFS.

4.1. *Proof of Theorem 1.* The sufficiency of condition (2.2) follows from the inequality

$$|\Psi(f, x)| \leq V_\Psi(1, x) \|f\|.$$

The necessity of (2.2) is proved by way of contradiction. Suppose that (2.2) is not satisfied. Then

$$(4.1.1) \quad \limsup_{x \rightarrow \infty} V_\Psi(1, x) = \infty.$$

In view of (4.1.1), (2.1) and the properties of Ψ , it is possible to find by induction an increasing sequence (x_k) going to infinity and a sequence (g_k) of functions in \mathcal{M}_0 such that, if A_k is defined by $A_k = V_\Psi(1, x_k)$, then

$$(4.1.2) \quad A_1 \geq 16 \text{ and } A_k \geq 16 A_{k-1}, \quad k = 2, 3, \dots,$$

$$(4.1.3) \quad A_k \geq 16 \left(\sup_{x \in R^+} |\Psi(\sum_{i=1}^{k-1} \frac{g_i}{\sqrt{A_i}}, x)| \right)^2, \quad k = 2, 3, \dots,$$

and

$$(4.1.4) \quad |g_k| \leq 1, \quad |\Psi(g_k, x_k)| \geq \frac{3}{4} A_k, \quad k = 1, 2, \dots$$

Let

$$(4.1.5) \quad g(x) = \sum_{i=1}^{\infty} \frac{g_i(x)}{\sqrt{A_i}}.$$