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equivalent condition is that, for any compact subsets K_1, K_2 of X , the set of all $g \in G$ with $g(K_1) \cap K_2 \neq \emptyset$ is finite.

For a properly discontinuous action, the orbit space X/G is a Hausdorff space. For any $x \in X$, there exists a neighborhood U of x such that the (finite) set of all $g \in G$ with $gU \cap U \neq \emptyset$ equals the isotropy group $G_x = \{g \mid g \in G, g(x) = x\}$. If X is a normal complex space and G acts properly discontinuously by biholomorphic maps, then X/G is a normal complex space.

THEOREM. (H. Cartan [8], and [66] Exp. I). *If X is a bounded domain in \mathbf{C}^n , then the group \mathfrak{A} of all biholomorphic maps $X \rightarrow X$ with the topology of compact convergence is a Lie group. For compact subsets K_1, K_2 of X , the set of all $g \in \mathfrak{A}$ such that $gK_1 \cap K_2 \neq \emptyset$ is a compact subset of \mathfrak{A} . A subgroup of \mathfrak{A} is discrete if and only if it acts properly discontinuously.*

If X is a bounded symmetric domain, then a discrete subgroup Γ of \mathfrak{A} operates freely if and only if it has no elements of finite order.

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§ 1. THE HILBERT MODULAR GROUP AND THE EULER NUMBER OF ITS ORBIT SPACE

1.1. Let \mathfrak{H} be the upper half plane of all complex numbers with positive imaginary part. \mathfrak{H} is embedded in the complex projective line $\mathbf{P}_1\mathbf{C}$. A complex matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc \neq 0$ operates on $\mathbf{P}_1\mathbf{C}$ by

$$z \mapsto \frac{az + b}{cz + d}$$

The matrices with real coefficients and $ad - bc > 0$ carry \mathfrak{H} over into itself and constitute a group $\mathbf{GL}_2^+(\mathbf{R})$. The group

$$(1) \quad \mathbf{PL}_2^+(\mathbf{R}) = \mathbf{GL}_2^+(\mathbf{R}) / \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \neq 0 \right\}$$

operates effectively on \mathfrak{H} . As is well known, this is the group of all biholomorphic maps of \mathfrak{H} to itself.

Writing $z = x + iy$ ($x, y \in \mathbf{R}, y > 0$) we have on \mathfrak{H} the Riemannian metric

$$\frac{(dx)^2 + (dy)^2}{y^2}$$

which is invariant under the action of $\mathbf{PL}_2^+(\mathbf{R})$. The volume element equals $y^{-2} dx \wedge dy$.

We introduce the Gauß-Bonnet form

$$(2) \quad \omega = -\frac{1}{2\pi} \cdot \frac{dx \wedge dy}{y^2}$$

If Γ is a discrete subgroup of $\mathbf{PL}_2^+(\mathbf{R})$ acting freely on \mathfrak{H} and such that \mathfrak{H}/Γ is compact, then \mathfrak{H}/Γ is a compact Riemann surface of a certain genus p whose Euler number $e(\mathfrak{H}/\Gamma) = 2 - 2p$ is given by the formula

$$(3) \quad e(\mathfrak{H}/\Gamma) = \int_{\mathfrak{H}/\Gamma} \omega$$

We recall that the discrete subgroup Γ acts freely if and only if Γ has no elements of finite order.

1.2. Consider the n -fold cartesian product $\mathfrak{H}^n = \mathfrak{H} \times \dots \times \mathfrak{H}$. Let \mathfrak{A} be the group of all biholomorphic maps $\mathfrak{H}^n \rightarrow \mathfrak{H}^n$. The connectedness component of the identity of \mathfrak{A} equals the n -fold direct product of $\mathbf{PL}_2^+(\mathbf{R})$ with itself. We have an exact sequence

$$(4) \quad 1 \rightarrow \mathbf{PL}_2^+(\mathbf{R}) \times \dots \times \mathbf{PL}_2^+(\mathbf{R}) \rightarrow \mathfrak{A} \rightarrow S_n \rightarrow 1,$$

where S_n is the group of permutations of n objects corresponding here to the permutations of the n factors of \mathfrak{H}^n . The sequence (4) presents \mathfrak{A} as a semi-direct product. On \mathfrak{H}^n we use coordinates z_1, z_2, \dots, z_n with $z_k = x_k + iy_k$ and $y_k > 0$. We have a metric invariant under \mathfrak{A} :

$$\sum_{j=1}^n \frac{(dx_j)^2 + (dy_j)^2}{y_j^2}$$

The corresponding Gauß-Bonnet form ω is obtained by multiplying the forms belonging to the individual factors; see (2). Therefore

$$(5) \quad \omega = (-1)^n \cdot \frac{1}{(2\pi)^n} \frac{dx_1 \wedge dy_1}{y_1^2} \wedge \dots \wedge \frac{dx_n \wedge dy_n}{y_n^2}$$

If Γ is a discrete subgroup of \mathfrak{A} acting freely on \mathfrak{H}^n and such that \mathfrak{H}^n/Γ is compact, then \mathfrak{H}^n/Γ is a compact complex manifold whose Euler number is given by

$$(6) \quad e(\mathfrak{H}^n/\Gamma) = \int_{\mathfrak{H}^n/\Gamma} \omega.$$

$e(\mathfrak{H}^n/\Gamma)$ is always divisible by 2^n : for a compact complex n -dimensional manifold X we denote by $[X]$ the corresponding element in the complex cobordism group [58]. We have

$$(7) \quad [\mathfrak{H}^n/\Gamma] = 2^{-n} e(\mathfrak{H}^n/\Gamma) \cdot [(\mathbf{P}_1\mathbf{C})^n].$$

This follows, because the Chern numbers of \mathfrak{H}^n/Γ are proportional [37] to those of $(\mathbf{P}_1\mathbf{C})^n$. In particular, the Euler number and the arithmetic genus (Todd genus) of $(\mathbf{P}_1\mathbf{C})^n$ are 2^n and 1 respectively and thus $2^{-n} \cdot e(\mathfrak{H}^n/\Gamma)$ is the arithmetic genus of \mathfrak{H}^n/Γ .

1.3. We shall study special subgroups of the group of biholomorphic automorphisms of \mathfrak{H}^n . They are in fact discrete subgroups of $(\mathbf{PL}_2^+(\mathbf{R}))^n$. Let K be an algebraic number field of degree n over the field \mathbf{Q} of rational numbers. We assume K to be totally real, i.e., there are n different embeddings of K into the reals. We denote them by

$$K \rightarrow \mathbf{R}, \quad x \mapsto x^{(j)}, \quad x \in K$$

We may assume $x = x^{(1)}$. The element x is called totally positive (in symbols, $x \gg 0$) if all $x^{(j)}$ are positive. The group

$$\mathbf{GL}_2^+(K) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in K, ad - bc \gg 0 \right\}$$

acts on \mathfrak{H}^n as follows: for $z = (z_1, \dots, z_n) \in \mathfrak{H}^n$ we have

$$z_j \mapsto \frac{a^{(j)} z_j + b^{(j)}}{c^{(j)} z_j + d^{(j)}}.$$

The corresponding projective group

$$\mathbf{PL}_2^+(K) = \mathbf{GL}_2^+(K) / \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, a \in K^* \right\}$$

acts effectively on \mathfrak{S}^n . Thus $\mathbf{PL}_2^+(K) \subset (\mathbf{PL}_2^+(\mathbf{R}))^n$.

Let \mathfrak{o}_K be the ring of algebraic integers in K , then by considering only matrices with $a, b, c, d \in \mathfrak{o}_K$ and $ad - bc = 1$ we get the subgroup $\mathbf{SL}_2(\mathfrak{o}_K)$ of $\mathbf{GL}_2^+(K)$. The group $\mathbf{SL}_2(\mathfrak{o}_K) / \{1, -1\}$ is the famous Hilbert modular group. It is a discrete subgroup of $(\mathbf{PL}_2^+(\mathbf{R}))^n$. We shall denote it by $G(K)$ or simply by G , if no confusion can arise.

$$G = \mathbf{SL}_2(\mathfrak{o}_K) / \{1, -1\} \subset \mathbf{PL}_2^+(K) \subset (\mathbf{PL}_2^+(\mathbf{R}))^n$$

The Hilbert modular group was studied by Blumenthal [5]. An error of Blumenthal concerning the number of cusps was corrected by Maaß [53].

The quotient space \mathfrak{S}^n/G is not compact, but it has a finite volume with respect to the invariant metric. It is natural to use the Euler volume given in (5). The quotient space \mathfrak{S}^n/G is a complex space and not a manifold (for $n > 1$). We shall return to this point later. But the volume of \mathfrak{S}^n/G is well-defined and was calculated by Siegel ([72], [74]). The ζ -function of the field K enters. It is defined by

$$\zeta_K(s) = \sum_{\substack{\mathfrak{a} \in \mathfrak{o}_K \\ \mathfrak{a} \text{ an ideal}}} \frac{1}{N(\mathfrak{a})^s}.$$

This sum extends over all ideals in \mathfrak{o}_K , and $N(\mathfrak{a})$ denotes the norm of \mathfrak{a} . The series converges if the real part of the complex number s is greater than 1. It converges absolutely uniformly on any compact set contained in the half plane $\operatorname{Re}(s) > 1$. The function ζ_K can be holomorphically extended to $\mathbf{C} - \{1\}$. It has a pole of order 1 for $s = 1$. Let D_K denote the discriminant of the field K .

Then

$$(8) \quad D_K^{\frac{s}{2}} \cdot \pi^{-\frac{sn}{2}} \cdot \Gamma(s/2)^n \cdot \zeta_K(s)$$

is invariant under the substitution $s \rightarrow 1 - s$.

This is the well-known functional equation of $\zeta_K(s)$. It can be found in most books on algebraic number theory. See, for example, [52].

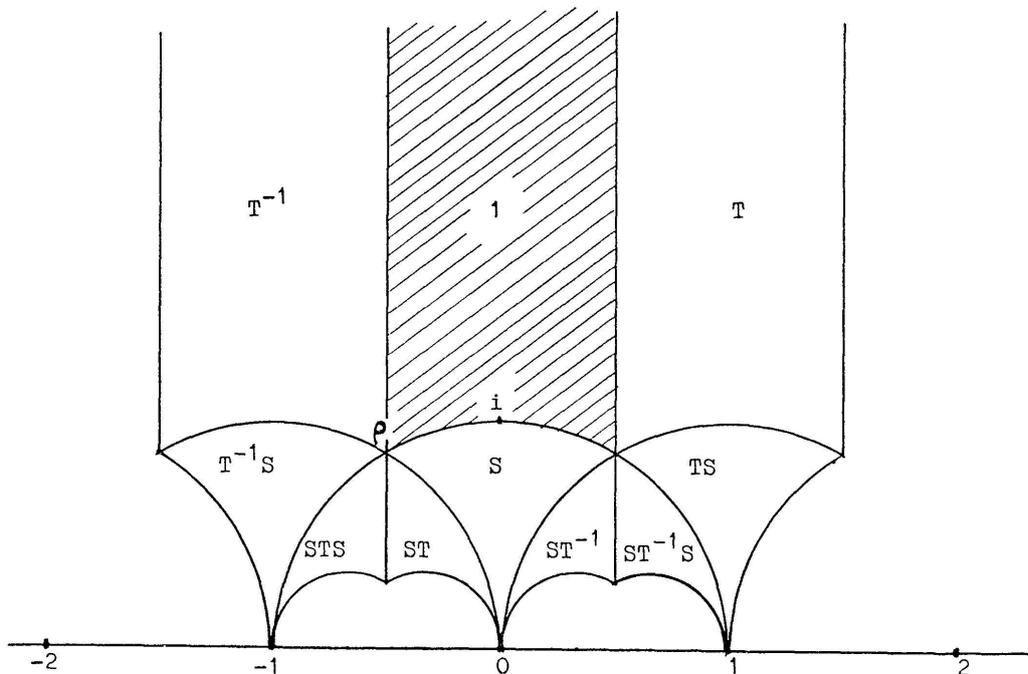
THEOREM (Siegel). *The Euler volume of \mathfrak{S}^n/G relates to the zeta-function as follows*

$$(9) \quad \int_{\mathfrak{H}^n/G} \omega = 2 \zeta_K(-1).$$

The formula (19) of [72] uses the volume element $\frac{dx_1 \wedge dy_1}{y_1^2} \wedge \dots \wedge \frac{dx_n \wedge dy_n}{y_n^2}$ and gives for the volume the value $2 \pi^{-n} \cdot D_K^{3/2} \zeta_K(2)$. If we multiply this value with $(-1)^n \cdot (2\pi)^{-n}$, we get $\int_{\mathfrak{H}^n/G} \omega$.

Formula (9) follows from the functional equation. It was pointed out by J. P. Serre [69] that such Euler volume formulas may be written more conveniently using values of the zeta functions at negative odd integers. $2\zeta_K(-1)$ is a rational number, a result going back to Hecke, see Siegel ([73] Ges. Abh. I, p. 546, [76]) and Klingen [44]. The rational number $2\zeta_K(-1)$ is in fact the rational Euler number of G in the sense of Wall [77], as we shall see later.

1.4. We shall write down explicit formulas for $2\zeta_K(-1)$ in some cases. For $K = \mathbf{Q}$, the group G is the ordinary modular group acting on \mathfrak{H} . A fundamental domain is described by the famous picture (see, for example, [68] p. 128).



The volume of \mathfrak{H}/G equals the volume of the shaded domain. By Siegel's general formula, the volume of the shaded domain with respect to $\frac{dx \wedge dy}{y^2}$ equals

$$2\pi^{-1} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi}{3}.$$

Therefore, we get for the Euler volume

$$(10) \quad \int_{\mathfrak{H}/G} \omega = -\frac{1}{6} = 2\zeta_{\mathbf{Q}}(-1).$$

We consider the real quadratic fields $K = \mathbf{Q}(\sqrt{d})$ where d is a square-free natural number > 1 . We recall that the discriminant D of K is given by

$$\begin{aligned} D &= 4d & \text{for } d \equiv 2,3 \pmod{4} \\ D &= d & \text{for } d \equiv 1 \pmod{4}. \end{aligned}$$

The ring \mathfrak{o}_K has additively the following \mathbf{Z} -bases.

$$\begin{aligned} \mathfrak{o}_K &= \mathbf{Z} + \mathbf{Z}\sqrt{d} & \text{for } d \equiv 2,3 \pmod{4} \\ \mathfrak{o}_K &= \mathbf{Z} + \mathbf{Z}\frac{1 + \sqrt{d}}{2} & \text{for } d \equiv 1 \pmod{4} \end{aligned}$$

THEOREM. *Let $K = \mathbf{Q}(\sqrt{d})$ be as above. Then for $d \equiv 1 \pmod{4}$*

$$(11) \quad 2\zeta_K(-1) = \frac{1}{15} \sum_{\substack{1 \leq b < \sqrt{d} \\ b \text{ odd}}} \sigma_1\left(\frac{d-b^2}{4}\right)$$

and for $d \equiv 2,3 \pmod{4}$

$$(12) \quad 2\zeta_K(-1) = \frac{1}{30}(\sigma_1(d) + 2 \cdot \sum_{1 \leq b < \sqrt{d}} \sigma_1(d-b^2))$$

where $\sigma_1(a)$ equals the sum of the divisors of a .

This theorem, though not exactly in this form, can be found in Siegel [76]. Compare also Gundlach [22], Zagier [78]. The κ_2 of Gundlach equals $4/\zeta_K(-1)$.

1.5. A reference for the following discussion is [71].

We always assume that Γ is a discrete subgroup of $(\mathbf{PL}^+(\mathbf{R}))^n$ and that \mathfrak{H}^n/Γ has finite volume.

Γ is irreducible if it contains no element $\gamma = (\gamma^{(1)}, \dots, \gamma^{(n)})$ such that $\gamma^{(i)} = 1$ for some i and $\gamma^{(j)} \neq 1$ for some j . See [71], p. 40 Corollary.

An element of $\mathbf{PL}_2^+(\mathbf{R})$ is parabolic if and only if it has exactly one fixed point in $\mathbf{P}_1\mathbf{C}$. This point belongs to $\mathbf{P}_1\mathbf{R} = \mathbf{R} \cup \infty$. An element $\gamma = (\gamma^{(1)}, \dots, \gamma^{(n)})$ of $(\mathbf{PL}_2^+(\mathbf{R}))^n$ is called parabolic if and only if all $\gamma^{(i)}$ are parabolic. The parabolic element γ has exactly one fixed point in $(\mathbf{P}_1\mathbf{C})^n$. It belongs to $(\mathbf{P}_1\mathbf{R})^n$. The parabolic points of Γ are by definition fixed points of the parabolic elements of Γ .

The above notation, hopefully, will not confuse the reader. The $\gamma^{(i)}$ are simply the components of the element γ of $(\mathbf{PL}_2^+(\mathbf{R}))^n$. If $\gamma \in \mathbf{PL}_2^+(K) \subset (\mathbf{PL}_2^+(\mathbf{R}))^n$ (compare 1.3), then, for γ represented by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the element $\gamma^{(i)}$ is represented by $\begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix}$ where $x \mapsto x^{(i)}$ is the i -th embedding of K in \mathbf{R} . For any group $\Gamma \subset (\mathbf{PL}_2^+(\mathbf{R}))^n$ we consider the orbits of parabolic points under the action of Γ on $(\mathbf{P}_1\mathbf{R})^n$. They are called parabolic orbits. Each such orbit consists only of parabolic points.

If Γ is irreducible, then there are only finitely many parabolic orbits. ([71], p. 46 Theorem 5).

Hereafter we shall assume in addition that Γ is irreducible.

If $x \in (\mathbf{P}_1\mathbf{R})^n$ is a parabolic point of Γ , we transform it to $\infty = (\infty, \dots, \infty)$ by an element ρ of $(\mathbf{PL}_2^+(\mathbf{R}))^n$, not necessarily belonging to Γ , of course. Thus $\rho x = \infty$.

Let Γ_x be the isotropy group of x .

$$\Gamma_x = \{ \gamma \mid \gamma \in \Gamma, \gamma x = x \}.$$

Then any element of $\rho\Gamma_x\rho^{-1}$ is of the form

$$(13) \quad z_j \mapsto \lambda^{(j)} z_j + \mu^{(j)}, \lambda^{(j)} > 0.$$

Consider the following multiplicative group

$$(14) \quad A = \{ t \mid t^{(i)} \in \mathbf{R}, t^{(i)} > 0, \prod_{j=1}^n t^{(j)} = 1 \}.$$

It is isomorphic to \mathbf{R}^{n-1} by taking logarithms. Each element of $\rho\Gamma_x\rho^{-1}$ (see (13)) satisfies $\lambda^{(1)} \cdot \lambda^{(2)} \dots \cdot \lambda^{(n)} = 1$, (compare [71], p. 43, Theorem 3). Therefore we have a natural homomorphism $\rho\Gamma_x\rho^{-1} \rightarrow A$ whose image is a discrete subgroup A_x of A of rank $n - 1$. The kernel consists of all the translations

$$z_j \mapsto z_j + \mu^{(j)}$$

where $\mu = (\mu^{(1)}, \dots, \mu^{(n)})$ belongs to a certain discrete subgroup M_x of \mathbf{R}^n of rank n . Thus we have an exact sequence

$$(15) \quad 0 \rightarrow M_x \rightarrow \rho \Gamma_x \rho^{-1} \rightarrow \Lambda_x \rightarrow 1.$$

Using the inner automorphisms of $\rho \Gamma_x \rho^{-1}$, the group Λ_x acts on M_x by componentwise multiplication. However, in the general case, (15) does not present $\rho \Gamma_x \rho^{-1}$ as a semi-direct product. For $n = 1$, the group Λ_x is trivial. For $n = 2$ it is infinite cyclic, $\rho \Gamma_x \rho^{-1}$ is a semi-direct product, and ρ can be chosen in such a way that $\rho \Gamma_x \rho^{-1}$ is exactly the group of all elements of the form (13) with $\lambda \in \Lambda_x$ and $\mu \in M_x$.

For any positive number d , the group $\rho \Gamma_x \rho^{-1}$ acts freely on

$$(16) \quad W = \{ z \mid z \in \mathfrak{H}^n, \prod_{j=1}^n \text{Im}(z_j) \geq d \}$$

where Im denotes the imaginary part. The orbit space $W/\rho \Gamma_x \rho^{-1}$ is a (non-compact) manifold with compact boundary

$$N = \partial W / \rho \Gamma_x \rho^{-1}.$$

Since ∂W is a principal homogeneous space for the semi-direct product $E = \mathbf{R}^n \rtimes \Lambda$ of all transformations

$$z_j \mapsto t^{(j)} z_j + a^{(j)}, t \in \Lambda, a \in \mathbf{R}^n$$

we can consider N as the quotient space of the group E (homeomorphic to \mathbf{R}^{2n-1}) by the discrete subgroup $\rho \Gamma_x \rho^{-1}$. Thus N is an Eilenberg-MacLane space. The $(2n-1)$ -dimensional manifold N is a torus bundle over the $(n-1)$ -dimensional torus Λ/Λ_x . The fibre is the n -dimensional torus \mathbf{R}^n/M_x , and N is obtained by the action of Λ_x on \mathbf{R}^n/M_x which is induced by the action $x_j \mapsto \lambda^{(j)} x_j + \mu^{(j)}$ of $\rho \Gamma_x \rho^{-1}$ on \mathbf{R}^n . Since, in general, $\mu^{(j)}$ is not necessarily an element of M_x , the action of Λ_x on \mathbf{R}^n/M_x need not be the one given by componentwise multiplication.

Definition ([71], p. 48). Let Γ be as before a discrete irreducible subgroup of $(\mathbf{PL}_2^+(\mathbf{R}))^n$ such that \mathfrak{H}^n/Γ has finite volume. Let x_ν ($1 \leq \nu \leq t$) be a complete set of Γ -inequivalent parabolic points of Γ . Choose elements $\rho_\nu \in (\mathbf{PL}_2^+(\mathbf{R}))^n$ with $\rho_\nu x_\nu = \infty$ and put $U_\nu = \rho_\nu^{-1}(W_\nu)$ where W_ν is defined as in (16) with some positive number d_ν instead of d . We say that Γ satisfies condition (F) if it admits (for some d_ν) a fundamental domain F of the form

$$F = F_0 \cup V_1 \cup \dots \cup V_t \quad (\text{disjoint union})$$

where F_0 is relatively compact in \mathfrak{H}^n and V_v is a fundamental domain of Γ_{x_v} in U_v .

The fundamental domain $F \subset \mathfrak{H}^n$ is by definition in one-to-one correspondence with \mathfrak{H}^n/Γ and V_v is in one-to-one correspondence with U_v/Γ_{x_v} .

The Hilbert modular group G of any totally real field K is a discrete irreducible subgroup of $(\mathbf{PL}_2^+(\mathbf{R}))^n$ with finite volume of \mathfrak{H}^n/G which satisfies condition (F). The existence of a fundamental domain with the required properties was shown by Blumenthal [5] as corrected by Maaß [53]. See Siegel [75] for a detailed exposition.

Two subgroups of $(\mathbf{PL}_2^+(\mathbf{R}))^n$ are called commensurable if their intersection is of finite index in both of them.

Any subgroup Γ of $(\mathbf{PL}_2^+(\mathbf{R}))^n$ which is commensurable with the Hilbert modular group G also satisfies (F).

We define

$$(17) \quad [G : \Gamma] = [G : (G \cap \Gamma)] / [\Gamma : (G \cap \Gamma)]$$

Then we get for the Euler volume

$$(18) \quad \int_{\mathfrak{H}^n/\Gamma} \omega = [G : \Gamma] \cdot \int_{\mathfrak{H}^n/G} \omega = [G : \Gamma] \cdot 2 \zeta_K(-1)$$

Remark. It is not known whether every discrete irreducible subgroup Γ of $(\mathbf{PL}_2^+(\mathbf{R}))^n$ such that \mathfrak{H}^n/Γ has finite volume satisfies Shimizu's condition (F).

Selberg has conjectured that any Γ satisfying (F) and having at least one parabolic point ($t \geq 1$) is conjugate in the group \mathfrak{A} of all automorphism of \mathfrak{H}^n to a group commensurable with the Hilbert modular group G of some totally real field K with $[K : \mathbf{Q}] = n$.

1.6. Harder [28] has proved a general theorem on the Euler number of not necessarily compact quotient spaces of finite volume. For the following result a direct proof can be given by the method used in [40].

THEOREM (Harder). *Let $\Gamma \subset (\mathbf{PL}_2^+(\mathbf{R}))^n$ be a discrete irreducible group satisfying condition (F) of the definition in 1.5. Suppose moreover*

that Γ operates freely on \mathfrak{H}^n . Then \mathfrak{H}^n/Γ is a complex manifold whose Euler number is given by

$$(19) \quad e(\mathfrak{H}^n/\Gamma) = \int_{\mathfrak{H}^n/\Gamma} \omega.$$

If Γ is commensurable with the Hilbert modular group G of K , (where K is a totally real field of degree n over \mathbf{Q}) then

$$(20) \quad e(\mathfrak{H}^n/\Gamma) = [G : \Gamma] \cdot 2\zeta_K(-1).$$

Proof. It follows from 1.5 that \mathfrak{H}^n/Γ contains a compact manifold Y with t boundary components $B_\nu = \partial W_\nu/\rho\Gamma_x\rho^{-1}$ (which are T^n -bundles over T^{n-1}). We have to choose the numbers d_ν sufficiently large. By the Gauß-Bonnet theorem of Allendoerfer-Weil-Chern [10]

$$e(\mathfrak{H}^n/\Gamma) = \int_Y \omega + \sum_{\nu=1}^t \int_{B_\nu} \prod$$

where \prod is a certain $(2n-1)$ -form. By the argument explained in [40], one can show easily that

$$\lim_{d_\nu \rightarrow \infty} \int_{B_\nu} \prod = 0. \text{ Q.E.D.}$$

Since the Hilbert modular group G always contains a subgroup Γ of finite index which operates freely and since \mathfrak{H}^n/Γ can be replaced up to homotopy by the compact manifold Y with boundary, $[G : \Gamma] \cdot 2\zeta_K(-1)$ is the Euler number of Γ in the sense of the rational cohomology theory of groups and thus $2\zeta_K(-1)$ is the Euler number of G in the sense of Wall [77].

THEOREM. Let $\Gamma \subset (\mathbf{PL}_2^+(\mathbf{R}))^n$ be a discrete irreducible group such that \mathfrak{H}^n/Γ has finite volume. Assume that Γ satisfies condition (F). The isotropy groups $\Gamma_z (z \in \mathfrak{H}^n)$ are finite cyclic and the set of those z with $|\Gamma_z| > 1$ projects down to a finite set in \mathfrak{H}^n/Γ . Thus \mathfrak{H}^n/Γ is a complex space with finitely many singularities. (For $n = 1$, these “branching points” are actually not singularities.)

Let $a_r(\Gamma)$ be the number of points in \mathfrak{H}^n/Γ which come from isotropy groups of order r . The Euler number of the space \mathfrak{H}^n/Γ is well-defined, and we have

$$(21) \quad e(\mathfrak{H}^n/\Gamma) = \int_{\mathfrak{H}^n/\Gamma} \omega + \sum_{r \geq 2} a_r(\Gamma) \frac{r-1}{r}.$$

The proof is an easy consequence of the Allendoerfer-Weil-Chern formula (compare [40], [65]).

The easiest example of (21) is of course the ordinary modular group $G = G(\mathbf{Q})$. We have $a_2(G) = a_3(G) = 1$ whereas the other $a_r(G)$ vanish. Thus

$$e(\mathfrak{H}/G) = -\frac{1}{6} + \frac{1}{2} + \frac{2}{3} = 1.$$

This checks, since \mathfrak{H}/G and \mathbf{C} are biholomorphically equivalent.

1.7. We shall apply (21) to the Hilbert modular group G and the extended Hilbert modular \hat{G} of a real quadratic field. \hat{G} is defined for any totally real field K . To define it we must say a few words about the units of K . They are the units of the ring \mathfrak{o}_K of algebraic integers. Let U be the group of these units. Its rank equals $n - 1$ by Dirichlet's theorem [6]. Let U^+ be the group of all totally positive units (see 1.3). It also has rank $n - 1$ because it contains $U^2 = \{\varepsilon^2 \mid \varepsilon \in U\}$.

The extended Hilbert modular group is defined as follows

$$\hat{G} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathfrak{o}_K, ad - bc \in U^+ \right\} / \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in U \right\}$$

We have an exact sequence

$$1 \rightarrow G \rightarrow \hat{G} \rightarrow U^+ / U^2 \rightarrow 1.$$

obtained by associating to each element of \hat{G} its determinant mod U^2 .

If $K = \mathbf{Q}(\sqrt{d})$ with d as in 1.4., then U^+ and U^2 are infinite cyclic groups and U^+ / U^2 is of order 2 or 1. The first case happens if and only if there is no unit in \mathfrak{o}_K with negative norm. If d is a prime p , then

$$U^+ \neq U^2 \Leftrightarrow p \equiv 3 \pmod{4}$$

$$U^+ = U^2 \Leftrightarrow p = 2 \text{ or } p \equiv 1 \pmod{4}.$$

Compare [30], Satz 133.

To apply (21) to the groups G and \hat{G} belonging to a real quadratic field we must know the numbers $a_r(G)$ and $a_r(\hat{G})$. They were determined by Gundlach [21] in some cases and in general by Prestel [61] using the idea that the isotropy groups G_z and \hat{G}_z respectively ($z \in \mathfrak{H}^2$) determine orders in imaginary extensions of K , which by an additional step relates

the $a_r(G)$ and $a_r(\hat{G})$ to ideal class numbers of quadratic imaginary fields over \mathbf{Q} . To write down Prestel's result we fix the following notation. A quadratic field k over \mathbf{Q} (real or imaginary) is completely given by its discriminant D . The class number of the field will be denoted by $h(D)$ or by $h(k)$.

Prestel has very explicit results for the Hilbert modular group G of any real quadratic field K and for the extended group \hat{G} in case the class number of K is odd. We shall indicate part of his result.

THEOREM. (Prestel). *Let d be squarefree, $d \geq 7$ and $(d, 6) = 1$. Let $K = \mathbf{Q}(\sqrt{d})$. Then for the Hilbert modular group $G(K)$ we have for*

$d \equiv 1 \pmod{4}$

$$a_2(G) = h(-4d), a_3(G) = h(-3d), a_r(G) = 0 \text{ for } r \neq 2, 3$$

and for $d \equiv 3 \pmod{8}$

$$a_2(G) = 10 \cdot h(-d), a_3(G) = h(-12d), a_r(G) = 0 \text{ for } r \neq 2, 3$$

and for $d \equiv 7 \pmod{8}$

$$a_2(G) = 4h(-d), a_3(G) = h(-12d), a_r(G) = 0 \text{ for } r \neq 2, 3$$

If d is a prime $\equiv 3 \pmod{4}$ and $d \neq 3$ we have for the extended group $\hat{G}(K)$ the following result:

If $d \equiv 3 \pmod{8}$, then

$$a_2(\hat{G}) = 3h(-d) + h(-8d), a_3(\hat{G}) = h(-12d)/2,$$

$$a_4(\hat{G}) = 4h(-d),$$

$$a_r(\hat{G}) = 0 \text{ for } r \neq 2, 3, 4.$$

If $d \equiv 7 \pmod{8}$, then

$$a_2(\hat{G}) = h(-d) + h(-8d), a_3(\hat{G}) = h(-12d)/2,$$

$$a_4(\hat{G}) = 2h(-d),$$

$$a_r(\hat{G}) = 0 \text{ for } r \neq 2, 3, 4.$$

Prestel gives the numbers $a_r(G)$ and $a_r(\hat{G})$ also for $d = 2, 3, 5$. For $d = 3$ we have

$$a_2(\hat{G}) = 3, a_3(\hat{G}) = 1, a_4(\hat{G}) = 1, a_{12}(\hat{G}) = 1,$$

all other $a_r(\hat{G}) = 0$.

We apply (12), (20) and (21) for $K = \mathbf{Q}(\sqrt{3})$ as an example

$$2\zeta_K(-1) = \frac{1}{30}(4 + 2\sigma_1(2)) = \frac{10}{30} = \frac{1}{3},$$

$$[G : \hat{G}] = \frac{1}{2},$$

$$e(H^2/\hat{G}) = \frac{1}{6} + 3 \cdot \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{11}{12} = 4.$$

We shall copy Prestel's table [61] of the $a_r(G)$ and the $a_r(\hat{G})$ (if known) for $K = \mathbf{Q}(\sqrt{d})$ up to $d = 41$. In [61] the table contains an error which was corrected in [62].

We also tabulate the values of $2\zeta_K(-1)$, $e(\mathfrak{H}^2/G)$, and of $e(\mathfrak{H}^2/\hat{G})$ if known. In the columns before $2\zeta_K(-1)$ we find the values of the $a_r(G)$; the values of the $a_r(\hat{G})$ are written behind $2\zeta_K(-1)$. If there is no entry, then the value is zero.

If the $a_r(\hat{G})$ and $e(\mathfrak{H}^2/\hat{G})$ are not given in the table, this means that either there exists a unit of negative norm and thus $G = \hat{G}$ or that the values are not known. This is indicated in the last column.

By Prestel $a_r(G) = 0$ for $r > 3$ and $K = \mathbf{Q}(\sqrt{d})$ with $d > 5$, and we have for $d > 5$

$$(22) \quad e(\mathfrak{H}^n/G) = 2\zeta_K(-1) + \frac{a_2(G)}{2} + a_3(G) \cdot \frac{2}{3}$$

Since the Euler number is an integer, we obtain by (11) and (12):

For $d > 5$, $d \equiv 1 \pmod{4}$, d square-free,

$$\sum_{\substack{1 \leq b < \sqrt{d} \\ b \text{ odd}}} \sigma_1\left(\frac{d-b^2}{4}\right) \equiv 0 \pmod{5}$$

For $d > 5$, $d \equiv 2, 3 \pmod{4}$, d square-free

$$\sigma_1(d) + 2 \sum_{1 \leq b < \sqrt{d}} \sigma_1(d-b^2) \equiv 0 \pmod{5}$$

Problem. Prove these congruences in the framework of elementary number theory.

d	2	3	4	5	6	$2\zeta_K(-1)$	2	3	4	6	12	$e(\mathfrak{S}^2/G)$	$e(\mathfrak{S}^2/\hat{G})$
2	2	2	2			1/6	—	—	—	—	—	4	$G = \hat{G}$
3	3	2			1	1/3	3	1	1		1	4	4
5	2	2		2		1/15	—	—	—	—	—	4	$G = \hat{G}$
6	6	3				1	5	1	2	1		6	6
7	4	4				4/3	5	2	2			6	6
10	6	4				7/3	—	—	—	—	—	8	$G = \hat{G}$
11	10	4				7/3	5	2	4			10	8
13	2	4				1/3	—	—	—	—	—	4	$G = \hat{G}$
14	12	4				10/3	8	2	4			12	10
15	8	6				4	—	—	—	—	—	12	?
17	4	2				2/3	—	—	—	—	—	4	$G = \hat{G}$
19	10	4				19/3	9	2	4			14	12
21	4	5				2/3	3	2		1		6	4
22	6	8				23/3	12	4	2			16	14
23	12	8				20/3	7	4	6			18	14
26	18	4				25/3	—	—	—	—	—	20	$G = \hat{G}$
29	6	6				1	—	—	—	—	—	8	$G = \hat{G}$
30	12	10				34/3	—	—	—	—	—	24	?
31	12	4				40/3	11	2	6			22	18
33	4	3				2	7	1		1		6	6
34	12	4				46/3	—	—	—	—	—	24	?
35	20	8				38/3	—	—	—	—	—	28	?
37	2	8				5/3	—	—	—	—	—	8	$G = \hat{G}$
38	18	8				41/3	16	4	6			28	22
39	16	10				52/3	—	—	—	—	—	40	?
41	8	2				8/3	—	—	—	—	—	8	$G = \hat{G}$

§ 2. THE CUSPS AND THEIR RESOLUTION
FOR THE 2-DIMENSIONAL CASE

2.1. Let K be a totally real algebraic field of degree n over \mathbf{Q} and M an additive subgroup of K which is a free abelian group of rank n . Such a group M is called a complete \mathbf{Z} -module of K . Let U_M^+ be the group of those units ε of K which are totally positive and satisfy $\varepsilon M = M$. Any $\alpha \in K$ with $\alpha M = M$ is automatically an algebraic integer and a unit.

The group U_M^+ is free of rank $n - 1$ (compare [6]).