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# ON IDEAL-ADIC TOPOLOGIES FOR A COMMUTATIVE RING

by Robert GILMER

Let  $R$  be a commutative ring and let  $A$  and  $B$  be ideals of  $R$  such that  $B \subseteq A$ . We wish to consider the relationship between the following two conditions:

- 1)  $R$  is complete in the  $A$ -adic topology <sup>1)</sup>.
- 2)  $R$  is complete in the  $B$ -adic topology.

Several results in this direction are known; for example:

**THEOREM 1.** ([4, Theorem 14, p. 275]) *Assume that  $R$  is Noetherian with identity, and that  $R$  is complete Hausdorff in its  $A$ -adic topology. Then  $R$  is complete in its  $B$ -adic topology.*

In [3], M. O'Malley proves the following theorem.

**THEOREM 2.** *If  $R$  has an identity, and if  $R$  is complete Hausdorff in its  $A$ -adic topology, then  $R$  is complete in the  $(b)$ -adic topology for each element  $b$  of  $A$ .*

In [2, Theorem 2.1 and Corollary 2.2], O'Malley extends his results in [3] to prove:

**THEOREM 3.** *If  $R$  contains an identity, if  $A = (a_1, \dots, a_n)$  is finitely generated, and if  $R$  is Hausdorff in the  $A$ -adic topology, then  $R$  is complete in its  $A$ -adic topology if and only if  $R$  is complete in its  $(a_i)$ -adic topology for each  $i$  between 1 and  $n$ .*

**COROLLARY 1.** *If  $R$  contains an identity, and if  $R$  is a complete Hausdorff space in its  $A$ -adic topology, then  $R$  is complete Hausdorff in its  $B$ -adic topology for each finitely generated ideal  $B$  contained in  $A$ .*

Moreover, O'Malley observes in [2] that Theorem 2, Theorem 3, and Corollary 1 are true without the assumption that  $R$  contains an identity, for the following result is valid.

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<sup>1)</sup> i.e. the topology for which a fundamental system of neighbourhoods of 0 is  $A, A^2, A^3, \dots$

PROPOSITION 1. Assume that  $R$  is a commutative ring, and  $S$  is a ring obtained by the canonical adjunction of an identity of characteristic zero to  $R$  (see [1, p. 4]). Let  $A$  be an ideal of  $R$ . Then  $A$  is an ideal of  $S$  and

1)  $R$  is Hausdorff in its  $A$ -adic topology if and only if  $S$  is Hausdorff in its  $A$ -adic topology;

2)  $R$  is complete in its  $A$ -adic topology if and only if  $S$  is complete in its  $A$ -adic topology.

O'Malley obtains the results we have cited from a much deeper theory of the set of  $R$ -endomorphisms of the power series ring  $R[[X]]$ . Our purpose here is to obtain O'Malley's results from basic topological considerations, independent of the theory of  $R$ -endomorphisms of  $R[[X]]$ .

PROPOSITION 2. Assume that  $\{A_i\}_{i=1}^n$  is a finite set of ideals of the commutative ring  $R$ , and let  $A = A_1 + \dots + A_n$ . If  $R$  is complete in its  $A_i$ -adic topology for each  $i$  between 1 and  $n$ , then  $R$  is complete in its  $A$ -adic topology.

*Proof.* We note that the  $A$ -adic topology on  $R$  is the topology induced by the sequence  $\{B_i\}_{i=1}^\infty$  of ideals, where  $B_i = A_1^i + \dots + A_n^i$ . This is true because  $A^i \supseteq B_i \supseteq A^{ni}$  for each positive integer  $i$ . Thus, if  $\{c_i\}_0^\infty$  is a Cauchy sequence in the  $A$ -adic topology, then by passage to a subsequence of  $\{c_i\}_0^\infty$ , we can assume that  $c_i - c_{i-1} \in B_i$  for each positive integer  $i$ . If we write  $c_i - c_{i-1} = a_{1i} + a_{2i} + \dots + a_{ni}$ , where  $a_{ji} \in A_j^i$ , then for each  $i$ ,

$$c_i = c_0 + \sum_{j=1}^n \sum_{k=1}^i a_{jk}.$$

The series  $\sum_{k=1}^\infty a_{jk}$  converges in the  $A_j$ -adic topology; we let  $a_j^* = \lim_k (a_{j1} + a_{j2} + \dots + a_{jk})$ . Then it is clear that the sequence  $\{c_i\}_0^\infty$  converges to  $c_0 + \sum_{j=1}^n a_j^*$  in the  $A$ -adic topology. Therefore  $R$  is complete in its  $A$ -adic topology.

We remark that in Proposition 2, the  $A$ -adic topology on  $R$  need not be Hausdorff, although the  $A_i$ -adic topology is Hausdorff for each  $i$ . For example, if  $k$  is a field, then  $k[[X, Y, Z]]/A$ , where  $A = (Z(1 - X - Y))$ , is complete Hausdorff under its  $[(X) + A]/A$ -adic and  $[(Y) + A]/A$ -adic topologies, but is not Hausdorff under its  $[(X, Y) + A]/A$ -adic topology.

THEOREM 4. Assume that  $R$  is a commutative ring, and that  $R$  is a complete Hausdorff space in its  $A$ -adic topology. If  $b \in A$ , then  $R$  is complete in its  $(b)$ -adic topology.

*Proof.* The  $(b)$ -adic topology on  $R$  is equivalent to the topology induced on  $R$  by the sequence  $\{B_i\}_{i=1}^{\infty}$  of ideals, where  $B_i = Rb^i$ . This is true since  $(b^i) \supseteq B_i \supseteq (b^{i+1})$  for each  $i$ . To prove that  $R$  is complete in its  $(b)$ -adic topology, it suffices to show that each sequence  $\{c_i\}_{i=0}^{\infty}$ , where  $c_i - c_{i-1} \in B_i$  for each  $i$ , converges in the  $(b)$ -adic topology. Since  $b \in A$ , the sequence  $\{c_i\}$  converges to an element  $c^*$  in the  $A$ -adic topology. We prove that  $c_i$  converges to  $c^*$  in the  $(b)$ -adic topology. Thus if  $c_i - c_{i-1} = r_i b^i$  for each positive integer  $i$ , then for positive integers  $k$  and  $n$ , we have

$$c_{k+n} - c_k = b^{k+1} [r_{k+1} + r_{k+2}b + \dots + r_{k+n}b^{n-1}].$$

Taking limits in the  $A$ -adic topology as  $n$  approaches infinity, and using the fact that the  $A$ -adic topology is Hausdorff, we obtain

$$c^* - c_k = b^{k+1} s_{k+1}^* \quad \text{where} \quad s_{k+1}^* = \sum_{n=1}^{\infty} r_{k+n} b^{n-1}.$$

It follows that  $c^* - c_k \in B_t$  for each  $k \geq t - 1$ , and  $\{c_i\}$  converges to  $c^*$  in the  $(b)$ -adic topology, as asserted.

Theorem 4 fails if the assumption that  $R$  is Hausdorff in the  $A$ -adic topology is dropped. For example, if  $R$  is idempotent, then  $R$  is complete in its  $R$ -adic topology, but  $R$  need not be complete in its  $(b)$ -adic topology for each  $b$  in  $R$ . For a less trivial example,  $Z \oplus Z$  is complete in its  $(Z \oplus (0))$ -adic topology, but not in its  $((2) \oplus (0))$ -adic topology.

Proposition 2 and Theorem 4 yield alternate proofs of Theorems 2 and 3 and Corollary 1 (dropping, in each case, the assumption that  $R$  has an identity).

We remark that in general,  $R$  need not be complete in its  $B$ -adic topology if  $R$  is complete Hausdorff in its  $A$ -adic topology, even if  $A$  is principal. Thus let  $D$  be an integral domain with identity containing a prime ideal  $C = (c_1, c_2, \dots, c_n, \dots)$  such that  $C$  is countably generated, but  $C$  is not the radical of a finitely generated ideal. (For example, let  $D = J[\{X_i\}_{i=1}^{\infty}]$ , where  $J$  is an integral domain with identity and let  $C = (\{X_i\}_{i=1}^{\infty})$ .) The ring  $R = D[[Y]]$  is a complete Hausdorff space in its  $(Y)$ -adic topology. But if  $B = (\{cY \mid c \in C\})$ , then  $R$  is not complete in the  $B$ -adic topology, for  $\{c_1 Y, c_1 Y + c_2^2 Y^2, \dots\}$  is a Cauchy sequence in the  $B$ -adic topology which converges to  $f = \sum_{i=1}^{\infty} c_i^i Y^i$  in the  $(Y)$ -adic topology. If this sequence converges in the  $B$ -adic topology, it must converge to  $f$ . But

$$f - (\sum_{i=1}^n c_i^i Y^i) = \sum_{i=n+1}^{\infty} c_i^i Y^i \notin B$$

for each positive integer  $n$ , for if  $\sum_{i=n+1}^{\infty} c_i^i Y^i \in B$ , then for some positive integer  $k$ ,  $\sum_{i=n+1}^{\infty} c_i^i Y^i \in (c_1 Y, \dots, c_k Y)$ , and  $c_i^i \in (c_1, \dots, c_k)$  for each  $i \geq n + 1$ .

It follows that  $C = \sqrt{(c_1, \dots, c_k)}$ , contrary to our assumptions concerning  $C$ .

*Added in proof.* Matthew O'Malley has pointed out to the author that in the remark preceding Theorem 4, the ring  $k[[X, Y, Z]]/A$  is Hausdorff in its  $[(X, Y) + A]/A$ -adic topology.

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