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ON IDEAL-ADIC TOPOLOGIES FOR A COMMUTATIVE RING

by Robert Gilmer

Let R be a commutative ring and let A and B be ideals of R such that $B \subseteq A$. We wish to consider the relationship between the following two conditions:

1) R is complete in the A-adic topology [1]).

2) R is complete in the B-adic topology.

Several results in this direction are known; for example:

Theorem 1. ([4, Theorem 14, p. 275]) *Assume that R is Noetherian with identity, and that R is complete Hausdorff in its A-adic topology. Then R is complete in its B-adic topology.*

In [3], M. O'Malley proves the following theorem.

Theorem 2. *If R has an identity, and if R is complete Hausdorff in its A-adic topology, then R is complete in the (b)-adic topology for each element b of A.*

In [2, Theorem 2.1 and Corollary 2.2], O'Malley extends his results in [3] to prove:

Theorem 3. *If R contains an identity, if $A = (a_1, ..., a_n)$ is finitely generated, and if R is Hausdorff in the A-adic topology, then R is complete in its A-adic topology if and only if R is complete in its (a_i)-adic topology for each i between 1 and n.*

Corollary 1. *If R contains an identity, and if R is a complete Hausdorff space in its A-adic topology, then R is complete Hausdorff in its B-adic topology for each finitely generated ideal B contained in A.*

Moreover, O'Malley observes in [2] that Theorem 2, Theorem 3, and Corollary 1 are true without the assumption that R contains an identity, for the following result is valid.

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[1]) i.e. the topology for which a fundamental system of neighbourhoods of 0 is A, A^2, A^3, ...

PROPOSITION 1. *Assume that R is a commutative ring, and S is a ring obtained by the canonical adjunction of an identity of characteristic zero to R (*see [1, p. 4]*). Let A be an ideal of R. Then A is an ideal of S and*

1) *R is Hausdorff in its A-adic topology if and only if S is Hausdorff in its A-adic topology;*

2) *R is complete in its A-adic topology if and only if S is complete in its A-adic topology.*

O'Malley obtains the results we have cited from a much deeper theory of the set of R-endomorphisms of the power series ring $R[[X]]$. Our purpose here is to obtain O'Malley's results from basic topological considerations, independent of the theory of R-endomorphisms of $R[[X]]$.

PROPOSITION 2. *Assume that $\{A_i\}_{i=1}^n$ is a finite set of ideals of the commutative ring R, and let $A = A_1 + \ldots + A_n$. If R is complete in its A_i-adic topology for each i between 1 and n, then R is complete in its A-adic topology.*

Proof. We note that the A-adic topology on R is the topology induced by the sequence $\{B_i\}_{i=1}^\infty$ of ideals, where $B_i = A_1^i + \ldots + A_n^i$. This is true because $A^i \supseteq B_i \supseteq A^{ni}$ for each positive integer i. Thus, if $\{c_i\}_0^\infty$ is a Cauchy sequence in the A-adic topology, then by passage to a subsequence of $\{c_i\}_0^\infty$, we can assume that $c_i - c_{i-1} \in B_i$ for each positive integer i. If we write $c_i - c_{i-1} = a_{1i} + a_{2i} + \ldots + a_{ni}$, where $a_{ji} \in A_j^i$, then for each i,

$$c_i = c_0 + \sum_{j=1}^n \sum_{k=1}^i a_{jk} \,.$$

The series $\sum_{k=1}^\infty a_{jk}$ converges in the A_j-adic topology; we let $a_j^* = \lim_k (a_{j1}+a_{j2}+ \ldots +a_{jk})$ Then it is clear that the sequence $\{c_i\}_0^\infty$ converges to $c_0 + \sum_{j=1}^n a_j^*$ in the A-adic topology. Therefore R is complete in its A-adic topology.

We remark that in Proposition 2, the A-adic topology on R need not be Hausdorff, although the A_i-adic topology is Hausdorff for each i. For example, if k is a field, then $k[[X, Y, Z]]/A$, where $A = (Z(1-X-Y))$, is complete Hausdorff under its $[(X) + A]/A$-adic and $[(Y) + A]/A$-adic topologies, but is not Hausdorff under its $[(X, Y) + A]/A$-adic topology.

THEOREM 4. *Assume that R is a commutative ring, and that R is a complete Hausdorff space in its A-adic topology. If $b \in A$, then R is complete in its (b)-adic topology.*

Proof. The (b)-adic topology on R is equivalent to the topology induced on R by the sequence $\{B_i\}_{i=1}^{\infty}$ of ideals, where $B_i = Rb^i$. This is true since $(b^i) \supseteq B_i \supseteq (b^{i+1})$ for each i. To prove that R is complete in its (b)-adic topology, it suffices to show that each sequence $\{c_i\}_{i=0}^{\infty}$, where $c_i - c_{i-1} \in B_i$ for each i, converges in the (b)-adic topology. Since $b \in A$, the sequence $\{c_i\}$ converges to an element c^* in the A-adic topology. We prove that c_i converges to c^* in the (b)-adic topology. Thus if $c_i - c_{i-1} = r_i b^i$ for each positive integer i, then for positive integers k and n, we have

$$c_{k+n} - c_k = b^{k+1}\left[r_{k+1} + r_{k+2}b + \ldots + r_{k+n}b^{n-1}\right].$$

Taking limits in the A-adic topology as n approaches infinity, and using the fact that the A-adic topology is Hausdorff, we obtain

$$c^* - c_k = b^{k+1}s_{k+1}^* \qquad \text{where} \qquad s_{k+1}^* = \textstyle\sum_{n=1}^{\infty} r_{k+n}b^{n-1}.$$

It follows that $c^* - c_k \in B_t$ for each $k \geqq t-1$, and $\{c_i\}$ converges to c^* in the (b)-adic topology, as asserted.

Theorem 4 fails if the assumption that R is Hausdorff in the A-adic topology is dropped. For example, if R is idempotent, then R is complete in its R-adic topology, but R need not be complete in its (b)-adic topology for each b in R. For a less trivial example, $Z \oplus Z$ is complete in its $(Z \oplus (0))$-adic topology, but not in its $((2) \oplus (0))$-adic topology.

Proposition 2 and Theorem 4 yield alternate proofs of Theorems 2 and 3 and Corollary 1 (dropping, in each case, the assumption that R has an identity).

We remark that in general, R need not be complete in its B-adic topology if R is complete Hausdorff in its A-adic topology, even if A is principal. Thus let D be an integral domain with identity containing a prime ideal $C = (c_1, c_2, \ldots, c_n, \ldots)$ such that C is countably generated, but C is not the radical of a finitely generated ideal. (For example, let $D = J[\{X_i\}_{i=1}^{\infty}]$, where J is an integral domain with identity and let $C = (\{X_i\}_{i=1}^{\infty})$.) The ring $R = D[[Y]]$ is a complete Hausdorff space in its (Y)-adic topology. But if $B = (\{cY \mid c \in C\})$, then R is not complete in the B-adic topology, for $\{c_1Y, c_1Y + c_2^2Y^2, \ldots\}$ is a Cauchy sequence in the B-adic topology which converges to $f = \sum_1^{\infty} c_i^i Y^i$ in the (Y)-adic topology. If this sequence converges in the B-adic topology, it must converge to f. But

$$f - \left(\textstyle\sum_1^n c_i^i Y^i\right) = \textstyle\sum_{n+1}^{\infty} c_i^i Y^i \notin B$$

for each positive integer, for if $\sum_{n+1}^{\infty} c_i^i Y^i \in B$, then for some positive integer k, $\sum_{n+1}^{\infty} c_i^i Y^i \in (c_1Y, \ldots, c_kY)$, and $c_i^i \in (c_1, \ldots, c_k)$ for each $i \geqq n+1$.

It follows that $C = \sqrt{(c_1, ..., c_k)}$, contrary to our assumptions concerning C.

Added in proof. Matthew O'Malley has pointed out to the author that in the remark preceding Theorem 4, the ring $k[[X, Y, Z]]/A$ is Hausdorff in its $[(X, Y) + A]/A$-adic topology.

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