Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 18 (1972)
Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON IDEAL-ADIC TOPOLOGIES FOR A COMMUTATIVE RING
Autor: Gilmer, Robert
DOI: https://doi.org/10.5169/seals-45370

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 27.03.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

ON IDEAL-ADIC TOPOLOGIES FOR A COMMUTATIVE RING

by Robert GILMER

Let R be a commutative ring and let A and B be ideals of R such that $B \subseteq A$. We wish to consider the relationship between the following two conditions:

1) R is complete in the A-adic topology [1]).

2) R is complete in the B-adic topology.

Several results in this direction are known; for example:

THEOREM 1. ([4, Theorem 14, p. 275]) *Assume that R is Noetherian with identity, and that R is complete Hausdorff in its A-adic topology. Then R is complete in its B-adic topology.*

In [3], M. O'Malley proves the following theorem.

THEOREM 2. *If R has an identity, and if R is complete Hausdorff in its A-adic topology, then R is complete in the (b)-adic topology for each element b of A.*

In [2, Theorem 2.1 and Corollary 2.2], O'Malley extends his results in [3] to prove:

THEOREM 3. *If R contains an identity, if $A = (a_1, ..., a_n)$ is finitely generated, and if R is Hausdorff in the A-adic topology, then R is complete in its A-adic topology if and only if R is complete in its (a_i)-adic topology for each i between 1 and n.*

COROLLARY 1. *If R contains an identity, and if R is a complete Hausdorff space in its A-adic topology, then R is complete Hausdorff in its B-adic topology for each finitely generated ideal B contained in A.*

Moreover, O'Malley observes in [2] that Theorem 2, Theorem 3, and Corollary 1 are true without the assumption that R contains an identity, for the following result is valid.

---

[1]) i.e. the topology for which a fundamental system of neighbourhoods of 0 is A, A^2, A^3, ...

PROPOSITION 1. *Assume that R is a commutative ring, and S is a ring obtained by the canonical adjunction of an identity of characteristic zero to R (*see [1, p. 4]*). Let A be an ideal of R. Then A is an ideal of S and*

1) *R is Hausdorff in its A-adic topology if and only if S is Hausdorff in its A-adic topology;*

2) *R is complete in its A-adic topology if and only if S is complete in its A-adic topology.*

O'Malley obtains the results we have cited from a much deeper theory of the set of R-endomorphisms of the power series ring $R[[X]]$. Our purpose here is to obtain O'Malley's results from basic topological considerations, independent of the theory of R-endomorphisms of $R[[X]]$.

PROPOSITION 2. *Assume that $\{A_i\}_{i=1}^n$ is a finite set of ideals of the commutative ring R, and let $A = A_1 + \ldots + A_n$. If R is complete in its A_i-adic topology for each i between* 1 *and n, then R is complete in its A-adic topology.*

Proof. We note that the A-adic topology on R is the topology induced by the sequence $\{B_i\}_{i=1}^\infty$ of ideals, where $B_i = A_1^i + \ldots + A_n^i$. This is true because $A^i \supseteq B_i \supseteq A^{ni}$ for each positive integer i. Thus, if $\{c_i\}_0^\infty$ is a Cauchy sequence in the A-adic topology, then by passage to a subsequence of $\{c_i\}_0^\infty$, we can assume that $c_i - c_{i-1} \in B_i$ for each positive integer i. If we write $c_i - c_{i-1} = a_{1i} + a_{2i} + \ldots + a_{ni}$, where $a_{ji} \in A_j^i$, then for each i,

$$c_i = c_0 + \sum_{j=1}^n \sum_{k=1}^i a_{jk}.$$

The series $\sum_{k=1}^\infty a_{jk}$ converges in the A_j-adic topology; we let $a_j^* = \lim_k (a_{j1} + a_{j2} + \ldots + a_{jk})$ Then it is clear that the sequence $\{c_i\}_0^\infty$ converges to $c_0 + \sum_{j=1}^n a_j^*$ in the A-adic topology. Therefore R is complete in its A-adic topology.

We remark that in Proposition 2, the A-adic topology on R need not be Hausdorff, although the A_i-adic topology is Hausdorff for each i. For example, if k is a field, then $k[[X, Y, Z]]/A$, where $A = (Z(1-X-Y))$, is complete Hausdorff under its $[(X) + A]/A$-adic and $[(Y) + A]/A$-adic topologies, but is not Hausdorff under its $[(X, Y) + A]/A$-adic topology.

THEOREM 4. *Assume that R is a commutative ring, and that R is a complete Hausdorff space in its A-adic topology. If $b \in A$, then R is complete in its (b)-adic topology.*

Proof. The (b)-adic topology on R is equivalent to the topology induced on R by the sequence $\{B_i\}_{i=1}^{\infty}$ of ideals, where $B_i = Rb^i$. This is true since $(b^i) \supseteq B_i \supseteq (b^{i+1})$ for each i. To prove that R is complete in its (b)-adic topology, it suffices to show that each sequence $\{c_i\}_{i=0}^{\infty}$, where $c_i - c_{i-1} \in B_i$ for each i, converges in the (b)-adic topology. Since $b \in A$, the sequence $\{c_i\}$ converges to an element c^* in the A-adic topology. We prove that c_i converges to c^* in the (b)-adic topology. Thus if $c_i - c_{i-1} = r_i b^i$ for each positive integer i, then for positive integers k and n, we have

$$c_{k+n} - c_k = b^{k+1}\left[r_{k+1} + r_{k+2}b + \ldots + r_{k+n}b^{n-1}\right].$$

Taking limits in the A-adic topology as n approaches infinity, and using the fact that the A-adic topology is Hausdorff, we obtain

$$c^* - c_k = b^{k+1}s^*_{k+1} \quad \text{where} \quad s^*_{k+1} = \sum_{n=1}^{\infty} r_{k+n}b^{n-1}.$$

It follows that $c^* - c_k \in B_t$ for each $k \geqq t - 1$, and $\{c_i\}$ converges to c^* in the (b)-adic topology, as asserted.

Theorem 4 fails if the assumption that R is Hausdorff in the A-adic topology is dropped. For example, if R is idempotent, then R is complete in its R-adic topology, but R need not be complete in its (b)-adic topology for each b in R. For a less trivial example, $Z \oplus Z$ is complete in its $(Z \oplus (0))$-adic topology, but not in its $((2) \oplus (0))$-adic topology.

Proposition 2 and Theorem 4 yield alternate proofs of Theorems 2 and 3 and Corollary 1 (dropping, in each case, the assumption that R has an identity).

We remark that in general, R need not be complete in its B-adic topology if R is complete Hausdorff in its A-adic topology, even if A is principal. Thus let D be an integral domain with identity containing a prime ideal $C = (c_1, c_2, \ldots, c_n, \ldots)$ such that C is countably generated, but C is not the radical of a finitely generated ideal. (For example, let $D = J[\{X_i\}_{i=1}^{\infty}]$, where J is an integral domain with identity and let $C = (\{X_i\}_{i=1}^{\infty})$.) The ring $R = D[[Y]]$ is a complete Hausdorff space in its (Y)-adic topology. But if $B = (\{cY \mid c \in C\})$, then R is not complete in the B-adic topology, for $\{c_1 Y, c_1 Y + c_2^2 Y^2, \ldots\}$ is a Cauchy sequence in the B-adic topology which converges to $f = \sum_1^{\infty} c_i^i Y^i$ in the (Y)-adic topology. If this sequence converges in the B-adic topology, it must converge to f. But

$$f - \left(\sum_1^n c_i^i Y^i\right) = \sum_{n+1}^{\infty} c_i^i Y^i \notin B$$

for each positive integer, for if $\sum_{n+1}^{\infty} c_i^i Y^i \in B$, then for some positive integer k, $\sum_{n+1}^{\infty} c_i^i Y^i \in (c_1 Y, \ldots, c_k Y)$, and $c_i^i \in (c_1, \ldots, c_k)$ for each $i \geqq n + 1$.

It follows that $C = \sqrt{(c_1, ..., c_k)}$, contrary to our assumptions concerning C.

Added in proof. Matthew O'Malley has pointed out to the author that in the remark preceding Theorem 4, the ring $k[[X, Y, Z]]/A$ is Hausdorff in its $[(X, Y) + A]/A$-adic topology.

REFERENCES

[1] GILMER, R. *Multiplicative Ideal Theory*, Queen's University, Kingston, Ontario, 1968.
[2] O'MALLEY, M. Some remarks on the formal power series ring, *Bull. Soc. Math. France*, 99 (1971), 247-258.
[3] —— On the Weierstrass preparation theorem, *Rocky Mt. J. Math.* (to appear).
[4] ZARISKI, O. and P. SAMUEL. *Commutative Algebra*, Vol. II, Van Nostrand, Princeton, N.J., 1960.

(Reçu le 20 janvier 1972)

Robert Gilmer
Florida State University
Tallahassee, Florida 32306