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A PROOF OF THE PRINCIPLE OF CIRCLE-TRANSFORMATION BY THE USE OF A THEOREM ON UNIVALENT FUNCTIONS

by Hiroshi HARUKI

The following theorem is well-known (see [2, p. 305]):

THEOREM A. Suppose that f is a meromorphic function of a complex variable z in $|z| < +\infty$. Then f is univalent if and only if f is a linear rational function of z .

The purpose of this note is to give a proof of the “only if” part of the following principle of circle-transformation of a linear rational function (see [1]) by the use of Theorem A:

Suppose that f (\neq const.) is a meromorphic function of z in $|z| < +\infty$. Then $w = f(z)$ transforms circles in the z -plane onto circles in the w -plane, including straight lines among circles, if and only if f is a linear rational function of z .

We now give a proof of the “only if” part of the above principle.

Let the domain where f is regular be D . We shall prove that f is univalent in $|z| < +\infty$. The proof is by contradiction. Assume contrary. Then there exist two distinct points a and b belonging to D such that

$$(1) \quad f(a) = f(b).$$

Let c be a point belonging to D such that $c \neq a$, $c \neq b$ and $f'(c) \neq 0$. Since $f \neq$ const., the existence of such c is guaranteed. Since $c \neq a$, $c \neq b$ and $f'(c) \neq 0$, there exists a circular neighborhood N of c satisfying the following three conditions:

(2) The closure of N lies entirely in D .

(3) The two points a and b are both exterior points of N .

(4) f is univalent in N .

Let C be the circumference of N and let the symmetric points of the two points a and b with respect to the circle C be a^* and b^* , respectively. By (3) a^* and b^* belong to N . By hypothesis $w = f(z)$ transforms circles in

the z -plane onto circles in the w -plane. Hence, by (2) $f(C)$ is not a straight line but a circle. Hence, by the Reflection Principle of Analytic Functions with respect to circles (see [2, p. 221]) the two points $f(a), f(a^*)$ and the two points $f(b), f(b^*)$ are symmetric, respectively, with respect to the circle $f(C)$ in the w -plane. So, by (1) we see that $f(a^*) = f(b^*)$. By (3) a^* and b^* belong to N . Since $a \neq b$, we have $a^* \neq b^*$. So, by (4) we have $f(a^*) \neq f(b^*)$, getting a contradiction.

Hence f is univalent in $|z| < +\infty$. Furthermore, by hypothesis f is meromorphic in $|z| < +\infty$. Hence, by Theorem A f is a linear rational function of z .

REFERENCES

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- [2] R. NEVANLINNA and V. PAATERO. *Introduction to complex analysis*, Addison-Wesley, 1964.

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