Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 18 (1972)

Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: PROOF OF THE PRINCIPLE OF CIRCLE-TRANSFORMATION BY THE USE OF A THEOREM ON UNIVALENT FUNCTIONS

Autor: Haruki, Hiroshi

DOI: https://doi.org/10.5169/seals-45365

A PROOF OF THE PRINCIPLE OF CIRCLE-TRANSFORMATION BY THE USE OF A THEOREM ON UNIVALENT FUNCTIONS

by Hiroshi HARUKI

The following theorem is well-known (see [2, p. 305]):

THEOREM A. Suppose that f is a meromorphic function of a complex variable z in $|z| < +\infty$. Then f is univalent if and only if f is a linear rational function of z.

The purpose of this note is to give a proof of the " only if " part of the following principle of circle-transformation of a linear rational function (see [1]) by the use of Theorem A:

Suppose that $f(\not\equiv \text{const.})$ is a meromorphic function of z in $|z| < +\infty$. Then $w = f(z)$ transforms circles in the z-plane onto circles in the w-plane, including straight lines among circles, if and only if f is a linear rational function of z.

We now give a proof of the " only if " part of the above principle.

Let the domain where f is regular be D. We shall prove that f is univalent in $|z| < +\infty$. The proof is by contradiction. Assume contrary. Then there exist two distinct points a and b belonging to D such that

(1) $f(a) = f(b)$.

Let c be a point belonging to D such that $c \neq a$, $c \neq b$ and $f'(c) \neq 0$. Since $f \not\equiv$ const., the existence of such c is guaranteed. Since $c \neq a$, $c \neq b$ and $f'(c) \neq 0$, there exists a circular neighborhood N of c satisfying the following three conditions:

(2) The closure of N lies entirely in D.

(3) The two points a and b are both exterior points of N.

(4) f is univalent in N.

Let C be the circumference of N and let the symmetric points of the two points a and b with respect to the circle C be a^* and b^*, respectively. By (3) a^* and b^* belong to N. By hypothesis $w = f(z)$ transforms circles in the z-plane onto circles in the w-plane. Hence, by (2) $f(C)$ is not a straight line but a circle. Hence, by the Reflection Principle of Analytic Functions with respect to circles (see [2, p. 221]) the tvo points $f(a), f(a^*)$ and the two points $f(b), f(b^*)$ are symmetric, respectively, with respect to the circle $f(C)$ in the w-plane. So, by (1) we see that $f(a^*) = f(b^*)$. By (3) a^* and b^* belong to N. Since $a \neq b$, we have $a^* \neq b^*$. So, by (4) we have $f(a^*) \neq f(b^*)$, getting a contradiction.

Hence f is univalent in $|z| < +\infty$. Furthermore, by hypothesis f is meromorphic in $|z| < +\infty$. Hence, by Theorem A f is a linear rational function of z.

REFERENCES

[1] Z. Nehari. *Conformal mapping*, McGraw-Hill, New York 1952, p. 160.
[2] R. Nevanlinna and V. Paatero. *Introduction to complex analysis*, Addison-Wesley, 1964.

(Reçu le 30 novembre 1971)

Hiroshi Haruki
Faculty of Mathematics
University of Waterloo
Waterloo, Ontario
Canada