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# A PROOF OF THE PRINCIPLE OF CIRCLE-TRANSFORMATION BY THE USE OF A THEOREM ON UNIVALENT FUNCTIONS

by Hiroshi HARUKI

The following theorem is well-known (see [2, p. 305]):

**THEOREM A.** Suppose that  $f$  is a meromorphic function of a complex variable  $z$  in  $|z| < +\infty$ . Then  $f$  is univalent if and only if  $f$  is a linear rational function of  $z$ .

The purpose of this note is to give a proof of the “only if” part of the following principle of circle-transformation of a linear rational function (see [1]) by the use of Theorem A:

Suppose that  $f$  ( $\neq$  const.) is a meromorphic function of  $z$  in  $|z| < +\infty$ . Then  $w = f(z)$  transforms circles in the  $z$ -plane onto circles in the  $w$ -plane, including straight lines among circles, if and only if  $f$  is a linear rational function of  $z$ .

We now give a proof of the “only if” part of the above principle.

Let the domain where  $f$  is regular be  $D$ . We shall prove that  $f$  is univalent in  $|z| < +\infty$ . The proof is by contradiction. Assume contrary. Then there exist two distinct points  $a$  and  $b$  belonging to  $D$  such that

$$(1) \quad f(a) = f(b).$$

Let  $c$  be a point belonging to  $D$  such that  $c \neq a$ ,  $c \neq b$  and  $f'(c) \neq 0$ . Since  $f \neq$  const., the existence of such  $c$  is guaranteed. Since  $c \neq a$ ,  $c \neq b$  and  $f'(c) \neq 0$ , there exists a circular neighborhood  $N$  of  $c$  satisfying the following three conditions:

(2) The closure of  $N$  lies entirely in  $D$ .

(3) The two points  $a$  and  $b$  are both exterior points of  $N$ .

(4)  $f$  is univalent in  $N$ .

Let  $C$  be the circumference of  $N$  and let the symmetric points of the two points  $a$  and  $b$  with respect to the circle  $C$  be  $a^*$  and  $b^*$ , respectively. By (3)  $a^*$  and  $b^*$  belong to  $N$ . By hypothesis  $w = f(z)$  transforms circles in

the  $z$ -plane onto circles in the  $w$ -plane. Hence, by (2)  $f(C)$  is not a straight line but a circle. Hence, by the Reflection Principle of Analytic Functions with respect to circles (see [2, p. 221]) the two points  $f(a), f(a^*)$  and the two points  $f(b), f(b^*)$  are symmetric, respectively, with respect to the circle  $f(C)$  in the  $w$ -plane. So, by (1) we see that  $f(a^*) = f(b^*)$ . By (3)  $a^*$  and  $b^*$  belong to  $N$ . Since  $a \neq b$ , we have  $a^* \neq b^*$ . So, by (4) we have  $f(a^*) \neq f(b^*)$ , getting a contradiction.

Hence  $f$  is univalent in  $|z| < +\infty$ . Furthermore, by hypothesis  $f$  is meromorphic in  $|z| < +\infty$ . Hence, by Theorem A  $f$  is a linear rational function of  $z$ .

#### REFERENCES

- [1] Z. NEHARI. *Conformal mapping*, McGraw-Hill, New York 1952, p. 160.
- [2] R. NEVANLINNA and V. PAATERO. *Introduction to complex analysis*, Addison-Wesley, 1964.

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