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COVERING BALLS FOR CURVES OF CONSTANT LENGTH

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An arc and a closed curve of length L in E^n are in particular sets of diameter at most L and $L/2$ respectively, and according to Jung's theorem [2] (see also [1]) they lie in (closed) balls of radii $[n/(2n+2)]^{1/2} L$ and $[n/(2n+2)]^{1/2} (L/2)$ respectively. We will prove that in fact each arc of length L lies in a ball of radius $L/2$ and each closed curve of length L lies in a ball of radius $L/4$, and no smaller balls will cover all such curves. Our arguments are entirely elementary.

In E^n, the *ball* of radius r centered at a point Z is $\{ X: XZ \leq r \}$; the *segment* $[PQ]$ is $\{ tP + (1-t) Q : 0 \leq t \leq 1 \}$ (so that in particular $[PP]$ is $\{ P \}$); the *midpoint* of a segment $[PQ]$ is the point $\frac{1}{2}(P+Q)$. A ball B *covers* a set Γ if there is a translation τ so that $\tau(\Gamma) \subseteq B$.

The argument depends on the following useful lemma.

LEMMA. Let P and Q be different points on a line l in E^n, and let M be the midpoint of the segment $[PQ]$. Then $XM \leq \frac{1}{2}(XP+XQ)$ for every point X of E^n, and the equality holds precisely for those points X on l that are not between P and Q.

Proof. Although an analytic proof is not difficult, we give a geometric argument. The assertions are easy to verify when X lies on l. Suppose that X is not on l and let Y be the point symmetric to X with respect to the midpoint M. Then $YP = XQ$, and it follows that

$$2XM = XM + MY = XY < XP + PY = XP + XQ$$

proving the lemma.

It is evident that the ball of radius $L/2$ centered at the middle point of an arc Γ of length L contains Γ. Another ball also works:

THEOREM 1. Let Γ be an arc of length L in E^n having endpoints P and Q, and let M be the midpoint of the segment $[PQ]$. Then the ball of radius $L/2$ centered at M contains Γ. If no smaller ball covers Γ, then Γ is a segment of length L.

Proof. Since $XM \leqq \frac{1}{2}(XP+XQ) \leqq L$ for any point X of Γ, the ball of radius $L/2$ centered at M surely contains Γ. Now suppose there is a point X on Γ such that $XM = L/2$. If $P = Q$, then Γ must be the segment $[PX]$ traversed twice; and a ball of radius $L/4$ covers Γ. So $P \neq Q$, and according to the lemma, X lies on the line through P and Q. This line meets the ball in two points, A and B. If both A and B lie on Γ, then Γ is the diameter $[AB]$. If only one of the points A and B lies on Γ, then Γ can be translated to lie entirely inside the ball; and so a smaller ball covers Γ. This proves the theorem.

Corollary. Let Γ be an arc of length L in E^n having endpoints P and Q, let M be the midpoint of $[PQ]$, and let N be the middle point of the arc Γ. Then every ball of radius $L/2$ centered at a point of $[MN]$ contains Γ.

Proof. If $B(Z)$ denotes the ball of radius $L/2$ centered at the point Z and if X is any point of $[MN]$, then $\Gamma \subseteq B(M) \cap B(N) \subseteq B(X)$.

The result for closed curves follows from the result for arcs.

THEOREM 2. Every closed curve Γ of length L in E^n lies in a ball of radius $L/4$. If no smaller ball covers Γ, then Γ is a segment of length $L/2$ traversed twice.

Proof. Let P be any point of Γ and let Q be the point of Γ at arc length $L/2$ from P. The points P and Q are the endpoints of two subarcs Γ_1 and Γ_2 of Γ both of which have length $L/2$. Let M be the midpoint of $[PQ]$. Then both Γ_1 and Γ_2 lie in the ball of radius $L/4$ centered at M by theorem 1, so surely this ball contains Γ. A point X of Γ such that $XM = L/4$ must lie on the line through P and Q, as in the proof of theorem 1. If both the points in which this line meets the bounding sphere of the ball lie on Γ, then evidently Γ is a diameter traversed twice. Otherwise a smaller ball covers Γ.

The approach to problems of this kind through this lemma is due to Amram Meir, who used it some years ago to show that every plane curve of length 1 lies in a semidisk of diameter 1 (a result that is in a sense sharper than theorem 1). Meir's result has not been published.

REFERENCES

[1] DANZER, L., B. GRÜNBAUM and V. KLEE, " Helly's theorem and its relatives ", *Proc. Sympos. Pure Math.*, vol. 7, Amer. Math. Soc., Providence, R. I., 1963, pp. 101-180.

[2] JUNG, H. W. E., " Über die kleinste Kugel, die eine räumliche Figur einschliesst ", *J. Reine Angew. Math. 123* (1901), pp. 241-257.

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Added in proof:

The 3-space case of theorem 2 has recently been proved analytically by J. C. C. Nitsche [3], and Meir's proof of the result mentioned in the last paragraph is reproduced in [4].

[3] NITSCHE, J. C. C., "The smallest sphere containing a rectificable curve", *Amer. Math. Monthly* 78 (1971), pp. 881-882.

[4] WETZEL, John E., "Sectorial covers for curves of constant length", to appear in *Canad. Math. Bull.*

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