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COVERING BALLS FOR CURVES OF CONSTANT LENGTH

by John E. WETZEL

An arc and a closed curve of length L in E^n are in particular sets of diameter at most L and $L/2$ respectively, and according to Jung's theorem [2] (see also [1]) they lie in (closed) balls of radii $[n/(2n+2)]^{1/2} L$ and $[n/(2n+2)]^{1/2} (L/2)$ respectively. We will prove that in fact each arc of length L lies in a ball of radius $L/2$ and each closed curve of length L lies in a ball of radius $L/4$, and no smaller balls will cover all such curves. Our arguments are entirely elementary.

In E^n , the *ball* of radius r centered at a point Z is $\{ X: XZ \leq r \}$; the *segment* $[PQ]$ is $\{ tP + (1-t)Q: 0 \leq t \leq 1 \}$ (so that in particular $[PP]$ is $\{ P \}$); the *midpoint* of a segment $[PQ]$ is the point $\frac{1}{2}(P+Q)$. A ball B covers a set Γ if there is a translation τ so that $\tau(\Gamma) \subseteq B$.

The argument depends on the following useful lemma.

LEMMA. Let P and Q be different points on a line l in E^n , and let M be the midpoint of the segment $[PQ]$. Then $XM \leq \frac{1}{2}(XP + XQ)$ for every point X of E^n , and the equality holds precisely for those points X on l that are not between P and Q .

Proof. Although an analytic proof is not difficult, we give a geometric argument. The assertions are easy to verify when X lies on l . Suppose that X is not on l and let Y be the point symmetric to X with respect to the midpoint M . Then $YP = XQ$, and it follows that

$$2XM = XM + MY = XY < XP + PY = XP + XQ$$

proving the lemma.

It is evident that the ball of radius $L/2$ centered at the middle point of an arc Γ of length L contains Γ . Another ball also works:

THEOREM 1. Let Γ be an arc of length L in E^n having endpoints P and Q , and let M be the midpoint of the segment $[PQ]$. Then the ball of radius $L/2$ centered at M contains Γ . If no smaller ball covers Γ , then Γ is a segment of length L .

Proof. Since $XM \leq \frac{1}{2}(XP + XQ) \leq L$ for any point X of Γ , the ball of radius $L/2$ centered at M surely contains Γ . Now suppose there is a point X on Γ such that $XM = L/2$. If $P = Q$, then Γ must be the segment $[PX]$ traversed twice; and a ball of radius $L/4$ covers Γ . So $P \neq Q$, and according to the lemma, X lies on the line through P and Q . This line meets the ball in two points, A and B . If both A and B lie on Γ , then Γ is the diameter $[AB]$. If only one of the points A and B lies on Γ , then Γ can be translated to lie entirely inside the ball; and so a smaller ball covers Γ . This proves the theorem.

Corollary. Let Γ be an arc of length L in E^n having endpoints P and Q , let M be the midpoint of $[PQ]$, and let N be the middle point of the arc Γ . Then every ball of radius $L/2$ centered at a point of $[MN]$ contains Γ .

Proof. If $B(Z)$ denotes the ball of radius $L/2$ centered at the point Z and if X is any point of $[MN]$, then $\Gamma \subseteq B(M) \cap B(N) \subseteq B(X)$.

The result for closed curves follows from the result for arcs.

THEOREM 2. Every closed curve Γ of length L in E^n lies in a ball of radius $L/4$. If no smaller ball covers Γ , then Γ is a segment of length $L/2$ traversed twice.

Proof. Let P be any point of Γ and let Q be the point of Γ at arc length $L/2$ from P . The points P and Q are the endpoints of two subarcs Γ_1 and Γ_2 of Γ both of which have length $L/2$. Let M be the midpoint of $[PQ]$. Then both Γ_1 and Γ_2 lie in the ball of radius $L/4$ centered at M by theorem 1, so surely this ball contains Γ . A point X of Γ such that $XM = L/4$ must lie on the line through P and Q , as in the proof of theorem 1. If both the points in which this line meets the bounding sphere of the ball lie on Γ , then evidently Γ is a diameter traversed twice. Otherwise a smaller ball covers Γ .

The approach to problems of this kind through this lemma is due to Amram Meir, who used it some years ago to show that every plane curve of length 1 lies in a semidisk of diameter 1 (a result that is in a sense sharper than theorem 1). Meir's result has not been published.

Vide-leer-empty

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Added in proof:

The 3-space case of theorem 2 has recently been proved analytically by J. C. C. Nitsche [3], and Meir's proof of the result mentioned in the last paragraph is reproduced in [4].

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