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# COVERING BALLS FOR CURVES OF CONSTANT LENGTH

by John E. WETZEL

An arc and a closed curve of length  $L$  in  $E^n$  are in particular sets of diameter at most  $L$  and  $L/2$  respectively, and according to Jung's theorem [2] (see also [1]) they lie in (closed) balls of radii  $[n/(2n+2)]^{1/2} L$  and  $[n/(2n+2)]^{1/2} (L/2)$  respectively. We will prove that in fact each arc of length  $L$  lies in a ball of radius  $L/2$  and each closed curve of length  $L$  lies in a ball of radius  $L/4$ , and no smaller balls will cover all such curves. Our arguments are entirely elementary.

In  $E^n$ , the *ball* of radius  $r$  centered at a point  $Z$  is  $\{X: XZ \leq r\}$ ; the *segment*  $[PQ]$  is  $\{tP + (1-t)Q: 0 \leq t \leq 1\}$  (so that in particular  $[PP]$  is  $\{P\}$ ); the *midpoint* of a segment  $[PQ]$  is the point  $\frac{1}{2}(P+Q)$ . A ball  $B$  *covers* a set  $\Gamma$  if there is a translation  $\tau$  so that  $\tau(\Gamma) \subseteq B$ .

The argument depends on the following useful lemma.

**LEMMA.** Let  $P$  and  $Q$  be different points on a line  $l$  in  $E^n$ , and let  $M$  be the midpoint of the segment  $[PQ]$ . Then  $XM \leq \frac{1}{2}(XP + XQ)$  for every point  $X$  of  $E^n$ , and the equality holds precisely for those points  $X$  on  $l$  that are not between  $P$  and  $Q$ .

*Proof.* Although an analytic proof is not difficult, we give a geometric argument. The assertions are easy to verify when  $X$  lies on  $l$ . Suppose that  $X$  is not on  $l$  and let  $Y$  be the point symmetric to  $X$  with respect to the midpoint  $M$ . Then  $YP = XQ$ , and it follows that

$$2XM = XM + MY = XY < XP + PY = XP + XQ$$

proving the lemma.

It is evident that the ball of radius  $L/2$  centered at the middle point of an arc  $\Gamma$  of length  $L$  contains  $\Gamma$ . Another ball also works:

**THEOREM 1.** Let  $\Gamma$  be an arc of length  $L$  in  $E^n$  having endpoints  $P$  and  $Q$ , and let  $M$  be the midpoint of the segment  $[PQ]$ . Then the ball of radius  $L/2$  centered at  $M$  contains  $\Gamma$ . If no smaller ball covers  $\Gamma$ , then  $\Gamma$  is a segment of length  $L$ .

*Proof.* Since  $XM \leq \frac{1}{2}(XP + XQ) \leq L$  for any point  $X$  of  $\Gamma$ , the ball of radius  $L/2$  centered at  $M$  surely contains  $\Gamma$ . Now suppose there is a point  $X$  on  $\Gamma$  such that  $XM = L/2$ . If  $P = Q$ , then  $\Gamma$  must be the segment  $[PX]$  traversed twice; and a ball of radius  $L/4$  covers  $\Gamma$ . So  $P \neq Q$ , and according to the lemma,  $X$  lies on the line through  $P$  and  $Q$ . This line meets the ball in two points,  $A$  and  $B$ . If both  $A$  and  $B$  lie on  $\Gamma$ , then  $\Gamma$  is the diameter  $[AB]$ . If only one of the points  $A$  and  $B$  lies on  $\Gamma$ , then  $\Gamma$  can be translated to lie entirely inside the ball; and so a smaller ball covers  $\Gamma$ . This proves the theorem.

*Corollary.* Let  $\Gamma$  be an arc of length  $L$  in  $E^n$  having endpoints  $P$  and  $Q$ , let  $M$  be the midpoint of  $[PQ]$ , and let  $N$  be the middle point of the arc  $\Gamma$ . Then every ball of radius  $L/2$  centered at a point of  $[MN]$  contains  $\Gamma$ .

*Proof.* If  $B(Z)$  denotes the ball of radius  $L/2$  centered at the point  $Z$  and if  $X$  is any point of  $[MN]$ , then  $\Gamma \subseteq B(M) \cap B(N) \subseteq B(X)$ .

The result for closed curves follows from the result for arcs.

**THEOREM 2.** Every closed curve  $\Gamma$  of length  $L$  in  $E^n$  lies in a ball of radius  $L/4$ . If no smaller ball covers  $\Gamma$ , then  $\Gamma$  is a segment of length  $L/2$  traversed twice.

*Proof.* Let  $P$  be any point of  $\Gamma$  and let  $Q$  be the point of  $\Gamma$  at arc length  $L/2$  from  $P$ . The points  $P$  and  $Q$  are the endpoints of two subarcs  $\Gamma_1$  and  $\Gamma_2$  of  $\Gamma$  both of which have length  $L/2$ . Let  $M$  be the midpoint of  $[PQ]$ . Then both  $\Gamma_1$  and  $\Gamma_2$  lie in the ball of radius  $L/4$  centered at  $M$  by theorem 1, so surely this ball contains  $\Gamma$ . A point  $X$  of  $\Gamma$  such that  $XM = L/4$  must lie on the line through  $P$  and  $Q$ , as in the proof of theorem 1. If both the points in which this line meets the bounding sphere of the ball lie on  $\Gamma$ , then evidently  $\Gamma$  is a diameter traversed twice. Otherwise a smaller ball covers  $\Gamma$ .

The approach to problems of this kind through this lemma is due to Amram Meir, who used it some years ago to show that every plane curve of length 1 lies in a semidisk of diameter 1 (a result that is in a sense sharper than theorem 1). Meir's result has not been published.

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*Added in proof:*

The 3-space case of theorem 2 has recently been proved analytically by J. C. C. Nitsche [3], and Meir's proof of the result mentioned in the last paragraph is reproduced in [4].

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