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define a parallelepiped  $\Pi^{(p)}$  in  $E^l$  which we shall call the  $p$ -th *pseudocompound* of the parallelepiped  $\Pi$  defined by (8.4).

*Remarks.* Mahler (1955) defined the  $p$ -th *compound* of any symmetric convex set, and the pseudocompound of a parallelepiped is closely related to its compound. But the compound of a parallelepiped is not necessarily a parallelepiped. Except for the notation, the  $(n-1)$ -st pseudocompound is the same as the dual of a parallelepiped, and hence the results of the last subsection may be interpreted as special cases of the results of the present subsection.

**THEOREM 8D** (Mahler 1955). *Let  $\lambda_1, \dots, \lambda_n$  and  $v_1, \dots, v_l$  be the successive minima of a parallelepiped  $\Pi$  and of its  $p$ -th pseudocompound  $\Pi^{(p)}$ , respectively. For  $\sigma \in C(n, p)$  put  $\lambda_\sigma = \prod_{i \in \sigma} \lambda_i$  and order the elements of  $C(n, p)$  as  $\sigma_1, \dots, \sigma_l$  such that  $\lambda_{\sigma_1} \leq \dots \leq \lambda_{\sigma_l}$ . Then*

$$v_j \gg \ll \lambda_{\sigma_j} \quad (j = 1, \dots, l).$$

*Moreover, if  $\mathbf{x}_1, \dots, \mathbf{x}_n$  are linearly independent integer points with (8.1), i.e. with  $|L_i(\mathbf{x}_j)| \leq \lambda_j R_i$  ( $i, j = 1, \dots, n$ ), and if for  $\tau = \{j_1, \dots, j_p\}$  in  $C(n, p)$  we put  $\mathbf{X}_\tau = \mathbf{x}_{j_1} \wedge \dots \wedge \mathbf{x}_{j_p}$ , then*

$$|L_\sigma^{(p)}(\mathbf{X}_\tau)| \ll \lambda_\tau R_\sigma \quad (\sigma, \tau \in C(n, p)).$$

## 9. OUTLINE OF THE PROOF OF THE THEOREMS ON SIMULTANEOUS APPROXIMATION TO ALGEBRAIC NUMBERS

**9.1.** Let us see what happens if we try to generalize Roth's proof to prove, say, Corollary 7B. In Roth's proof we constructed a polynomial  $P(x_1, \dots, x_m)$  in  $m$  variables  $x_1, \dots, x_m$  which had a zero of high order at  $(\alpha, \dots, \alpha)$ . Hence the natural thing to try would be

(a) to construct a polynomial  $P(x_{11}, \dots, x_{1l}; \dots; x_{m1}, \dots, x_{ml})$  in  $ml$  variables of total degree  $\leq r_h$  in each block of variables  $x_{h1}, \dots, x_{hl}$  ( $h = 1, \dots, m$ ) with a zero of high order at  $(\alpha_1, \dots, \alpha_l; \dots; \alpha_1, \dots, \alpha_l)$ . Then

(b) one would have to show that if each of  $m$  given rational  $l$ -tuples  $\left(\frac{p_{h1}}{q_h}, \dots, \frac{p_{hl}}{q_h}\right)$  ( $h = 1, \dots, m$ ) satisfies (7.2), then  $P$  also has a zero of high order at

$$\left( \frac{p_{11}}{q_1}, \dots, \frac{p_{1l}}{q_1}; \dots; \frac{p_{m1}}{q_m}, \dots, \frac{p_{ml}}{q_m} \right).$$

Finally

(c) one would have to show that under suitable conditions  $P$  cannot have a high zero at such a rational point.

If we proceed in this fashion, we encounter difficulties in (c). In Roth's Lemma 3C it was essential that  $P$  had rather different degrees in its variables and that the denominators in  $\frac{p_1}{q_1}, \dots, \frac{p_m}{q_m}$  increased very fast. In our present situation the first  $l$  denominators are equal, so that Roth's Lemma does not apply. The example  $m = 1, l = 2, P(x_1, x_2) = (x_1 - x_2)^r$  shows that we cannot expect to have a lemma similar to Roth's in our present context, since  $P$  has a zero of order as high as  $r$  at every point  $(\xi, \xi)$ .

The polynomial  $P$  is defined on  $E^l \times \dots \times E^l$  ( $m$  copies). While it is difficult to say much about the order of vanishing of  $P$  at rational points  $\mathbf{r}_1 \times \dots \times \mathbf{r}_m$ , it is easier to show that  $P$  cannot have a zero of high order on certain linear manifolds  $\mathcal{M}_1 \times \dots \times \mathcal{M}_m$  where each  $\mathcal{M}_h$  is a rational (i.e. defined by a linear equation with rational coefficients) hyperplane in  $E^l$ . We can illustrate this when  $m = 1$ . Namely,  $\mathcal{M}_1$  is defined by an equation  $a_0 + a_1x_1 + \dots + a_lx_l = 0$  which can be normalized such that  $a_0, a_1, \dots, a_l$  are coprime rational integers. If  $P(x_1, \dots, x_l)$  has a zero of order  $\geq i$  on  $\mathcal{M}_1$  (i.e.  $P$  has a zero of order  $\geq i$  at every point of  $\mathcal{M}_1$ ), then  $P(x_1, \dots, x_l) = (a_0 + a_1x_1 + \dots + a_lx_l)^i R(x_1, \dots, x_l)$ , where  $R$  has integer coefficients by Gauss' Lemma. It follows that

$$(9.1) \quad (H(M))^i \leq H(P)$$

where  $H(M)$  is the height of  $M(\mathbf{x}) = a_0 + a_1x_1 + \dots + a_lx_l$ . This inequality provides a good upper bound for  $i$  if  $H(M)$  is large.

**9.2.** It will be more convenient to deal with hyperplanes through the origin in  $E^{l+1}$  than with hyperplanes in  $E^l$ . Hence we shall put

$$(9.2) \quad n = l + 1$$

and we shall consider polynomials  $P(x_{11}, \dots, x_{1n}; \dots; x_{m1}, \dots, x_{mn})$  which are homogeneous of degree  $r_h$  in each block of variables  $x_{h1}, \dots, x_{hn}$  ( $h = 1, \dots, m$ ). The manifold  $\mathcal{M}_1 \times \dots \times \mathcal{M}_m$  now becomes a subspace defined by  $L_1(x_{11}, \dots, x_{1n}) = \dots = L_m(x_{m1}, \dots, x_{mn}) = 0$ , where each  $L_h$  is a not

identically vanishing linear form in  $x_{h1}, \dots, x_{hm}$  ( $h=1, \dots, m$ ). The polynomial  $P$  vanishes on  $\mathcal{M}_1 \times \dots \times \mathcal{M}_m$  precisely if it lies in the ideal generated by  $L_1, \dots, L_m$ . A suitable definition of the index is now as follows.

Let  $L_h = L_h(x_{h1}, \dots, x_{hm})$  ( $h=1, \dots, m$ ) be not identically vanishing linear forms. For positive integers  $r_1, \dots, r_m$  and for  $c \geq 0$  let  $\mathcal{T}(c)$  be the ideal generated by the products  $L_1^{i_1} \dots L_m^{i_m}$  with

$$\frac{i_1}{r_1} + \dots + \frac{i_m}{r_m} \geq c.$$

The index of  $P$  with respect to  $(L_1, \dots, L_m; r_1, \dots, r_m)$  is the largest value of  $c$  such that  $P \in \mathcal{T}(c)$  if  $P$  is not identically zero, and it is  $+\infty$  if  $P$  is identically zero.

**9.3.** Now suppose that  $L(\mathbf{x}) = \alpha_1 x_1 + \dots + \alpha_n x_n$  has real algebraic coefficients. In analogy with Lemma 3A in step (a) in the proof of Roth's Theorem, one can construct a polynomial  $P$  as above which is not identically zero and which has not too large rational integer coefficients, such that  $P$  has index at least

$$\left(\frac{1}{n} - \varepsilon\right)m,$$

with respect to  $(L, \dots, L; r_1, \dots, r_m)$ . Here  $L$  really occurs with  $m$  different meanings; namely, the  $h$ -th copy of  $L$  means  $\alpha_1 x_{h1} + \dots + \alpha_n x_{hn}$  ( $h=1, \dots, m$ ). Perhaps it should be explained why the factor  $\frac{1}{2} - \varepsilon$  in Lemma 3A is now

replaced by  $\frac{1}{n} - \varepsilon$ . A form  $P$  in  $mn$  variables  $x_{11}, \dots, x_{1n}; \dots; x_{m1}, \dots, x_{mn}$

is also a form in  $L, x_{12}, \dots, x_{1n}; \dots; L, x_{m2}, \dots, x_{mn}$  provided  $\alpha_1 \neq 0$  (and where  $L$  occurs with different meanings again). Now for "most" monomials

in  $L, x_{12}, \dots, x_{1n}; \dots; L, x_{m2}, \dots, x_{mn}$  the degree in  $L$  will be about  $\frac{1}{n}$  times

the total degree of the monomial, and hence will be greater than  $\left(\frac{1}{n} - \varepsilon\right)$  times the total degree of the monomial.

But a result with only one linear form  $L$  is not enough. In general, say when dealing with General Roth Systems, one has  $n$  linear forms  $L_1, \dots, L_n$  to start with, and one can deal with them simultaneously. The following result now replaces Lemma 3A.



LEMMA 9A. Let  $L_1, \dots, L_n$  be not identically vanishing linear forms with real algebraic coefficients. Suppose  $\varepsilon > 0$ . Then if  $m > m_0(L_1, \dots, L_n; \varepsilon)$  and if  $r_1, \dots, r_m$  are positive integers, there is a polynomial  $P(x_{11}, \dots, x_{1n}; \dots; x_{m1}, \dots, x_{mn}) \not\equiv 0$  with rational integer coefficients such that

- (i)  $P$  is homogeneous in  $x_{h1}, \dots, x_{hn}$  of degree  $r_h$  ( $h=1, \dots, m$ ).
- (ii)  $P$  has index  $\geq \left(\frac{1}{n} - \varepsilon\right)m$  with respect to  $(L_i, \dots, L_i; r_1, \dots, r_m)$  ( $i=1, \dots, n$ ).
- (iii)  $H(P) \leq B^{r_1 + \dots + r_m}$  where  $B = B(L_1, \dots, L_m)$ .

This takes care of generalizing part (a) of Roth's proof. We have chosen our definition of the index such that (c) has a chance of going through, and in fact one can derive from Roth's Lemma 3C a more general lemma that applies in our situation. Namely, if  $M_1(\mathbf{x}), \dots, M_m(\mathbf{x})$  are linear forms with rational integer coefficients, then under suitable conditions the index of  $P$  with respect to  $(M_1, \dots, M_m; r_1, \dots, r_m)$  is  $\leq \varepsilon$ .

**9.4.** If thus remains to deal with part (b). Suppose, say, that we want to derive a criterion for General Roth Systems as defined in §7.3. Suppose  $L_1, \dots, L_n$  are linear forms with real algebraic coefficients and suppose  $\gamma_1 + \dots + \gamma_n = 0$ . Suppose there is a  $\delta > 0$  and there are arbitrarily large values of  $Q$  for which there is an integer point  $\mathbf{x} \neq \mathbf{0}$  with  $|L_i(\mathbf{x})| < Q^{\gamma_i - \delta}$  ( $i=1, \dots, n$ ). Assume in particular that this is true for  $Q = Q_1, \dots, Q_m$  and with integer points  $\mathbf{x}_1, \dots, \mathbf{x}_m$ , respectively. An argument like the one used in the proof of Lemma 3B shows that if suitable auxiliary conditions are satisfied, then the polynomial  $P$  of Lemma 9A does in fact have

$$P(\mathbf{x}_1, \dots, \mathbf{x}_m) = 0.$$

But this is not what we really need. Namely, we need a rational subspace of the type  $\mathcal{M}_1 \times \dots \times \mathcal{M}_m$  where each  $\mathcal{M}_h$  is a hyperplane of  $E^n$ , such that  $P$  vanishes on this subspace.

There is a way out of this difficulty, although it is a rather costly one. Namely, we have to assume that for each  $Q_h$  ( $h=1, \dots, m$ ) there is not just one but there are

$$l = n - 1$$

linearly independent integer points  $\mathbf{x}_h^{(1)}, \dots, \mathbf{x}_h^{(l)}$  with

$$(9.3) \quad |L_i(\mathbf{x}_h^{(j)})| \leq Q_h^{\gamma_i - \delta} \quad (i=1, \dots, n; j=1, \dots, l; h=1, \dots, m).$$

Now if  $\mathcal{M}_h$  is the hyperplane through  $\mathbf{0}$  spanned by  $\mathbf{x}_h^{(1)}, \dots, \mathbf{x}_h^{(l)}$  ( $h=1, \dots, m$ ), then one can show that  $P$  vanishes on  $\mathcal{M}_1 \times \dots \times \mathcal{M}_m$ . In fact one can show that if  $M_h$  is the linear form defining  $\mathcal{M}_h$  ( $h=1, \dots, m$ ), then the index of  $P$  with respect to  $(M_1, \dots, M_m; r_1, \dots, r_m)$  is  $\geq m\epsilon$ , which in conjunction with (c) gives the desired contradiction.

**9.5.** But what have we really shown now? The inequalities

$$(9.4) \quad |L_i(\mathbf{x})| \leq Q^{\gamma_i} \quad (i=1, \dots, n)$$

define a parallelepiped. The presence of  $l = n - 1$  linearly independent integer points  $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(l)}$  with  $|L_i(\mathbf{x}^{(j)})| \leq Q^{\gamma_i - \delta}$  ( $i=1, \dots, n; j=1, \dots, l$ ) means that the  $(n-1)$ st minimum  $\lambda_{n-1} = \lambda_{n-1}(Q)$  satisfies  $\lambda_{n-1} \leq Q^{-\delta}$ . The inequalities (9.3) mean precisely that  $\lambda_{n-1}(Q) \leq Q^{-\delta}$  for  $Q = Q_1, Q_2, \dots, Q_m$ . Thus we obtain a theorem about  $\lambda_{n-1}$ :

**THEOREM 9B.** (*Theorem on the next to last minimum*). Suppose  $n \geq 2$  and  $L_1, \dots, L_n$  are linearly independent linear forms with real algebraic coefficients, and suppose  $L_1^*, \dots, L_n^*$  are their duals. Suppose  $\delta > 0$ , suppose  $\gamma_1 + \dots + \gamma_n = 0$ , and let  $\Sigma$  be the set of integers  $i$  in  $1 \leq i \leq n$  for which

$$\gamma_i + \delta \geq 0.$$

There is a  $Q_0 = Q_0(L_1, \dots, L_n; \gamma_1, \dots, \gamma_n; \delta)$  with the following property: Let  $\lambda_1 = \lambda_1(Q), \dots, \lambda_n = \lambda_n(Q)$  be the successive minima of the parallelepiped  $\Pi(Q)$  given by (9.4). Then for  $Q > Q_0$  either

$$(9.5) \quad \lambda_{n-1} > Q^{-\delta}$$

or

$$(9.6) \quad L_i^*(\mathbf{x}_n^*) = 0 \text{ for every } i \in \Sigma,$$

where  $\mathbf{x}_1^*, \dots, \mathbf{x}_n^*$  are the duals <sup>1)</sup> to linearly independent integer points  $\mathbf{x}_1, \dots, \mathbf{x}_n$  with  $\mathbf{x}_j \in \lambda_j \Pi$  ( $j=1, \dots, n$ ).

It was clear from the discussion above that some inequality such as (9.5) would result. The hyperplanes  $\mathcal{M}$  of the discussion above were spanned by  $\mathbf{x}_1, \dots, \mathbf{x}_{n-1}$  (but with the notation  $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(l)}$ ), and hence the coefficients

<sup>1)</sup> I.e. they satisfy  $\mathbf{x}_i \mathbf{x}_j^* = \delta_{ij}$  ( $i, j=1, \dots, n$ ).

in the defining equation for  $\mathcal{M}$  are proportional to  $\mathbf{x}_n^*$ . The alternative (9.6) had to be put in to allow for the possibility that  $\mathcal{M}$  behaves in a somewhat degenerate fashion. In most cases, e.g., if the coefficients of some  $L_i^*$  with  $i \in \Sigma$  are linearly independent over the rationals, then no integer point  $\mathbf{x} \neq \mathbf{0}$  can satisfy (9.6), and then (9.5) must hold.

Theorem 9B gives information on  $\lambda_{n-1}$  rather than on  $\lambda_1$ . In what follows, transference theorems will be used to gain information on  $\lambda_1$ .

**9.6.** Theorem 9B says that if  $Q$  is large and  $\lambda_{n-1} < Q^{-\delta}$ , then  $\mathbf{x}_n^*$  must lie in a certain subspace. The inequality (8.7) of Mahler's Theorem 8C further restricts the possibilities for  $\mathbf{x}_n^*$ . A combination of these results yields

**COROLLARY 9C.** *Suppose  $L_1, \dots, L_n, \gamma_1, \dots, \gamma_n, \delta, \mathbf{x}_1 = \mathbf{x}_1(Q), \dots, \mathbf{x}_n = \mathbf{x}_n(Q), \mathbf{x}_1^* = \mathbf{x}_1^*(Q), \dots, \mathbf{x}_n^* = \mathbf{x}_n^*(Q)$  are as above. Suppose there are arbitrarily large values of  $Q$  with*

$$(9.7) \quad \lambda_{n-1} < Q^{-\delta}.$$

*Then there is a fixed vector  $\mathbf{c}$  and there are arbitrarily large values of  $Q$  with (9.7) and with  $\mathbf{x}_n^*(Q) = \mathbf{c}$ .*

Next, the condition (9.7) will be replaced by

$$(9.8) \quad \lambda_{n-1} < Q^{-\delta} \lambda_n.$$

The latter condition usually is milder, since  $\lambda_n \gg 1$  by (8.5).

**THEOREM 9D.** *(Theorem on the last two minima). Suppose  $L_1, \dots, L_n, \gamma_1, \dots, \gamma_n, \delta, \mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{x}_1^*, \dots, \mathbf{x}_n^*$  are as above. Suppose there are arbitrarily large values of  $Q$  with (9.8). Then there are arbitrarily large values of  $Q$  with (9.8) and with  $\mathbf{x}_n^*(Q) = \mathbf{c}$ , where  $\mathbf{c}$  is a fixed vector.*

To prove this theorem one needs Davenport's Lemma (Theorem 8B). Namely, put  $\rho_0 = (\lambda_1 \dots \lambda_{n-2} \lambda_{n-1}^2)^{1/n}$  and

$$\rho_1 = \rho_0 / \lambda_1, \dots, \rho_{n-1} = \rho_0 / \lambda_{n-1}, \text{ but } \rho_n = \rho_0 / \lambda_{n-1}.$$

By Davenport's Lemma we can compare the successive minima  $\lambda_1, \dots, \lambda_n$  of  $\Pi$  with the successive minima  $\lambda'_1, \dots, \lambda'_n$  of another parallelepiped  $\Pi'$ . We have  $\lambda'_j \gg \ll \rho_j \lambda_j$  ( $j=1, \dots, n$ ) and  $\rho_0 \ll \lambda'_1 \ll \dots \ll \lambda'_{n-1} \ll \rho_0 \ll (\lambda_{n-1} / \lambda_n)^{1/n} \ll Q^{-\delta/n}$  by (8.5) and (9.8). Hence  $\lambda'_{n-1} < Q^{-\delta/(2n)}$  if  $Q$  is large, and applying Corollary 9C to  $\Pi'$  we see that  $\mathbf{x}_n^{*'}(Q)$  is the same

for arbitrarily large values of  $Q$ , which in turn (by the last assertion of Davenport's Lemma) implies that  $\mathbf{x}_n^*(Q)$  is the same for certain arbitrarily large values of  $Q$ .

**9.7. THEOREM 9E. (Subspace Theorem).** Suppose  $L_1, \dots, L_n, \gamma_1, \dots, \gamma_n, \delta, \mathbf{x}_1(Q), \dots, \mathbf{x}_n(Q)$  are as above. Suppose there is a  $d$  in  $1 \leq d \leq n-1$  such that

$$(9.9) \quad \lambda_d < \lambda_{d+1} Q^{-\delta}$$

for certain arbitrarily large values of  $Q$ . Then there is a fixed rational subspace  $S^d$  of dimension  $d$  such that for some arbitrarily large values of  $Q$  with (9.9), the points

$$\mathbf{x}_1(Q), \dots, \mathbf{x}_d(Q) \text{ lie in } S^d.$$

For the proof put  $p = n - d$  and construct the linear forms  $L_\sigma^{(p)}$  as in §8.4. Also put  $\Gamma_\sigma = \sum_{i \in \sigma} \gamma_i$ . The inequalities

$$|L_\sigma^{(p)}(\mathbf{X})| \leq Q^{\Gamma_\sigma} \quad (\sigma \in C(n, p))$$

define the  $p$ -th pseudocompound  $\Pi^{(p)}$  of  $\Pi$ . By Mahler's Theorem 8D the last two minima  $v_{l-1}, v_l$  of this pseudocompound have

$$v_{l-1} \gg \ll \lambda_d \lambda_{d+2} \lambda_{d+3} \dots \lambda_n, \quad v_l \gg \ll \lambda_{d+1} \lambda_{d+2} \lambda_{d+3} \dots \lambda_n,$$

whence  $v_{l-1} < v_l Q^{-\delta/2}$  for large  $Q$  by (9.9). An application of Theorem 9D shows that  $\mathbf{X}_l^{*-1}$  is the same for some arbitrarily large values of  $Q$ . Some algebra combined with the last assertion of Theorem 8D shows that (because of (9.9))  $\mathbf{X}_l^*$  is proportional to  $\mathbf{x}_{d+1}^* \wedge \dots \wedge \mathbf{x}_n^*$ . It follows that the subspace  $S^*$  spanned by  $\mathbf{x}_{d+1}^*, \dots, \mathbf{x}_n^*$  is the same for some arbitrarily large values of  $Q$ . But for these values of  $Q$  the vectors  $\mathbf{x}_1, \dots, \mathbf{x}_d$  lie in the orthogonal complement  $S^d$  of  $S^*$ .

**9.8.** We shall illustrate the power of the Subspace Theorem by deducing Theorem 7E. Suppose we have  $\delta > 0$ ,  $1 \leq m < n$ ,  $m$  linearly independent linear forms  $L_1, \dots, L_m$  with real algebraic coefficients, and infinitely many integer solutions  $\mathbf{x} \neq \mathbf{0}$  of

<sup>1)</sup>  $\mathbf{X}_l^*$  in  $E^l$  is defined in terms of  $\Pi^{(p)}(Q)$  just as  $\mathbf{x}_n^*$  in  $E^n$  was defined in terms of  $\Pi(Q)$ .

$$|L_i(\mathbf{x})| \leq |\mathbf{x}|^{-((n-m)/m)-\delta} \quad (i = 1, \dots, m).$$

We may assume without loss of generality that  $L_1, \dots, L_m, x_1, \dots, x_{n-m}$  are linearly independent. Put  $L_{m+1}(\mathbf{x}) = x_1, \dots, L_n(\mathbf{x}) = x_{n-m}$ . It is easy to see that there is a  $\delta' > 0$  and there are arbitrarily large values of  $Q$  for which there are solutions  $\mathbf{x} \neq \mathbf{0}$  of

$$|L_i(\mathbf{x})| \leq Q^{\gamma_i - \delta'} \quad (i = 1, \dots, n)$$

where  $\gamma_1 = \dots = \gamma_m = -(n-m)/m$  and  $\gamma_{m+1} = \dots = \gamma_n = 1$ . For these values of  $Q$  one has  $\lambda_1 = \lambda_1(Q) < Q^{-\delta'}$ . Since  $\lambda_1 \leq \dots \leq \lambda_n$  and  $1 \ll \lambda_1 \dots \lambda_n \ll 1$ , there is a  $d$  with  $1 \leq d \leq n-1$  and a  $\delta'' > 0$  such that

$$(9.10) \quad \lambda_d < \lambda_{d+1} Q^{-\delta''}$$

for arbitrarily large values of  $Q$ . Let  $S^d$  be the subspace in the conclusion of Theorem 9E.

Let  $\Pi^*(Q)$  be the intersection of  $\Pi(Q)$  and  $S^d$ ; this is a symmetric convex set in  $S^d$ . Let  $\lambda_1^*, \dots, \lambda_d^*$  be the successive minima of  $\Pi^*(Q)$  with respect to the lattice  $\Lambda$  of integer points in  $S^d$ , and let  $V^* = V^*(Q)$  be the ( $d$ -dimensional) volume of  $\Pi^*(Q)$ . By applying (8.3) to the lattice  $\Lambda$  we obtain

$$(9.11) \quad 1 \ll \lambda_1^* \dots \lambda_d^* V^* \ll 1,$$

where the constants in  $\ll$  may depend on  $S^d$ . There are arbitrarily large values of  $Q$  for which  $\mathbf{x}_1(Q), \dots, \mathbf{x}_d(Q)$  lie in  $S^d$ , and for these values we have  $\lambda_1 = \lambda_1^*, \dots, \lambda_d = \lambda_d^*$ , whence by (8.5) and (9.10),

$$\begin{aligned} \lambda_1^* \dots \lambda_d^* &= \lambda_1 \dots \lambda_d = (\lambda_1 \dots \lambda_d)^{d/n} (\lambda_1 \dots \lambda_d)^{(n-d)/n} \\ &< (\lambda_1 \dots \lambda_d)^{d/n} (\lambda_{d+1} \dots \lambda_n)^{d/n} Q^{-\delta''d(n-d)/n} \ll Q^{-\delta''d(n-d)/n} = Q^{-\eta}, \end{aligned}$$

say. In conjunction with (9.11) this yields  $V^* \gg Q^\eta$ .

Now if  $L_1, \dots, L_m$  have rank  $r$  on  $S^d$ , then

$$V^* \ll Q^{-(r(n-m)/m)+d-r} = Q^{d-(rn/m)}.$$

It follows that  $d - (rn/m) \geq \eta > 0$  and that

$$r < dm/n.$$

This cannot happen if (7.6) holds, and hence  $L_1, \dots, L_m$  is a Roth System in this case. Since the case of linearly dependent forms  $L_1, \dots, L_m$  is trivial and since the other half of the theorem was proved in §7.3, Theorem 7E is established.