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# ON THE EXISTENCE AND THE REGULARITY OF SOLUTIONS OF LINEAR PSEUDO-DIFFERENTIAL EQUATIONS [1]

by Lars HÖRMANDER

## INTRODUCTION

Let $X$ be an open set in $\mathbf{R}^n$ (or a $C^\infty$ manifold) and let $P$ be a differential operator in $X$ with $C^\infty$ coefficients. Our purpose is to study the equation

$$Pu = f$$

where $u$ and $f$ are functions or distributions in $X$. Somewhat vaguely we can state the questions to be considered as follows:

*a)* What are the conditions on $P$ and on $X$ for a local or a global existence theorem to be valid?

*b)* What are the relations between the singularities of $u$ and those of $f$ when $X$ and $P$ are given?

In fact, these questions are so closely related that they can be considered as different forms of the same problem.

We shall look for answers in terms of geometric properties of the characteristics. These are defined as follows. If $P$ is of order $m$ and $u$, $\varphi \in C^\infty(X)$, then

$$P(e^{i\omega\varphi}u) = e^{i\omega\varphi}\left(\omega^m p(x, \operatorname{grad}\varphi)\, u + \omega^{m-1} Lu + \ldots\right)$$

where $p$ is a homogeneous function of degree $m$ on the cotangent bundle $T^*(X)$, called the characteristic polynomial or principal part (symbol) of $P$, and $L$ is a first order differential operator depending on $\varphi$. The zeros of $p$ in $T^*(X)\backslash 0$ are called (real) characteristics. The Hamilton-Jacobi integration theory for the characteristic equation $p(x, \operatorname{grad}\varphi) = 0$ also introduces certain curves in the level surfaces of $\varphi$, the bicharacteristics (see section 3.1). The classical methods used to relate geometrical and wave

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optics (or classical and wave mechanics) consist in first choosing a solution of the equation $p(x, \text{grad}\,\varphi) = 0$ and then determining $u$ as a solution of the (transport) equation $Lu = 0$, which is a first order equation along the bicharacteristics. This makes $P(e^{i\omega\varphi}u)$ of order $\omega^{m-2}$. There is also a much more refined method due to Luneburg (see Kline-Kay [1]), in which $u$ is replaced by an asymptotic series $\sum_0^\infty u_j\omega^{-j}$ in $\omega$. By successive choice of the functions $u_j$ one can obtain an asymptotic solution of the equation $P(ue^{i\omega\varphi}) = 0$. The importance of these expansions for the general theory of partial differential equations has been realized for quite some time (see Lax [1]) but it is only during the last few years that one has started to exploit them systematically. An important step in this direction was taken by Egorov [1] who also called attention to the results of Maslov [1]. A rigorous and systematic exposition incorporating the theory of pseudo-differential operators has been undertaken in Hörmander [6, 9, 10] and we shall discuss some of the results in Chapter II. Related ideas will be applied to differential equations with constant coefficients in Chapter I.

When discussing singularities we shall not only be concerned with their location but also with their local harmonic analysis. For a distribution in $X$ this leads to a set in the cosphere bundle of $X$. In the case of hyperfunctions the advantages of such a point of view were first pointed out by Sato [1]. Many variations of this idea are possible, and we shall indicate some (sections 1.6 and 2.2). The sets obtained by harmonic analysis of the singularities will be called wave front sets here. Since the geometrical objects associated with $P$, such as the characteristics, usually live in the cotangent bundle of $X$, it is natural to expect that they can be more easily related to the wave front set than to the set of singularities.

Since much work is in progress on the topics discussed in these lectures it seems useless to try to give a complete picture of the present state of the theory, but we do attempt to indicate the most important directions of this work. For references to some topics not discussed at all here see also Hörmander [13]. Originally we had planned to include a number of results concerning operators with variable coefficients involving the principal and the subprincipal part, that is, an invariantly defined function of degree one less than the principal part. However, these were left out because of the already excessive length of the survey, and we content ourselves with referring to Hörmander [5] (second order hypoelliptic differential equations), Radkevič [1], [2] (simplifications and extensions of these results), a forthcoming book on hypoelliptic operators by O. A. Olejnik and E. V. Radkevič; Mizohata-Ohya [1] and Flaschka-Strang [1] (hyperbolic operators with characteristics of constant multiplicity). The methods discussed in Chapter III can obviously be used to push much further in this direction. For the constant coefficient case a model result is given by Theorem 1.5.1.

# Chapter I

## Operators with constant coefficients

### 1.1. *Fundamental solutions*

A differential operator with constant coefficients in $\mathbf{R}^n$ can be written in the form $P(D)$ where $P$ is a polynomial in $n$ variables with complex coefficients and $D = (-i\partial/\partial x_1, \ldots, -i\partial/\partial x_n)$. Explicitly

$$P(D) = \sum a_\alpha D^\alpha$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index and the sum is finite.

It is easy to show that the equation

$$P(D)u = f \tag{1.1.1}$$

can always be solved locally. To do so we assume first that $f \in C_0^\infty$. If $u$ is a solution of (1.1.1) with a well defined Fourier transform $\hat{u}$, we must have $P(\xi)\,\hat{u}(\xi) = \hat{f}(\xi)$, and so by Fourier's inversion formula

$$u(x) = (2\pi)^{-n} \int e^{i<x,\xi>} \hat{f}(\xi)/P(\xi)\, d\xi\,. \tag{1.1.2}$$

However, $P$ may have zeros in or near $\mathbf{R}^n$ and this makes it necessary to deform the integration contour in order to obtain a well defined solution from (1.1.2).

First note that if $\Phi \in C_0^\infty(\mathbf{C}^n)$ and

$$\Phi(e^{i\theta}\zeta) = \Phi(\zeta)\,,\ \theta \in \mathbf{R}\,,\ \int \Phi(\zeta)\, d\lambda(\zeta) = 1\,, \tag{1.1.3}$$

where $d\lambda$ is the Lebesgue measure in $\mathbf{C}^n$, then

$$\int F(\zeta)\, \Phi(\zeta)\, d\lambda(\zeta) = F(0) \tag{1.1.4}$$

for any entire analytic function $F$. In fact, by Cauchy's integral formula

$$\int F(\zeta e^{i\theta})\, d\theta = 2\pi F(0)\,,$$

and if we multiply by $\Phi(\zeta)$ and integrate, (1.1.3) gives (1.1.4).

Let *Pol* $(m)$ be the complex vector space of polynomials of degree $\leqq m$ and let $Pol^0$ $(m)$ be the vector space with the origin removed. If $\Omega$ is a neighborhood of 0 in $\mathbf{C}^n$ one can find a $C^\infty$ map $\Phi : Pol^0(m) \to C_0^\infty(\Omega)$ which is homogeneous of degree zero, such that the range consists of functions satisfying (1.1.3) and for some constant $C$

$$(1.1.5) \quad \sum | Q^{(\alpha)}(0) | \leqq C | Q(\zeta) | , \quad Q \in Pol^0(m) , \quad \zeta \in \operatorname{supp} \Phi(Q) .$$

Here $Q^{(\alpha)}(\xi) = (iD)^\alpha Q$; the left hand side is of course a norm in *Pol* $(m)$. For a fixed $Q$ the existence of such a $\Phi$ is quite obvious for we can find $\theta \in \mathbf{R}^n$ such that $Q(z\theta) \neq 0$ when $|z| = 1$, and (1.1.5) is then fulfilled if the support of $\Phi$ is near this circle. The same $\Phi$ can be used for all $Q$ near by, and since functions satisfying (1.1.3) form a convex set the construction of $\Phi$ can be finished by means of a partition of unity in the set of all $Q$ with $\sum | Q^{(\alpha)}(0) | = 1$.

We now replace (1.1.2) by the expression

$$(1.1.6) \quad (Ef)(x) = (2\pi)^{-n} \int d\xi \int e^{i<x,\xi+\zeta>} \hat{f}(\xi+\zeta)/P(\xi+\zeta)\, \Phi(P_\xi,\zeta)\, d\lambda(\zeta)$$

where $P_\xi$ is the polynomial $\zeta \to P(\xi+\zeta)$. Since some derivative of $P$ is a constant, the function

$$(1.1.7) \qquad \tilde{P}(\xi) = \sum | P^{(\alpha)}(\xi) |$$

has a positive lower bound. Hence it follows from (1.1.5) that $P$ is bounded away from 0 in the support of the integrand, so (1.1.6) is well defined for $f \in C_0^\infty$. Differentiation under the integral sign gives $P(D)\, Ef = f$ in view of (1.1.4) and Fourier's inversion formula. Hence we have solved (1.1.1) when $f \in C_0^\infty(\mathbf{R}^n)$. The map $f \to Ef$ commutes with translations so there is a distribution which we also denote by $E$ for which $Ef = E * f$. Since $(P(D)\, E) * f = f$ for all $f \in C_0^\infty$ we have $P(D)\, E = \delta$, the Dirac measure at 0. To solve (1.1.1) for arbitrary $f \in \mathscr{E}'(\mathbf{R}^n)$, the space of distributions with compact support, it is therefore sufficient to choose $u = E * f$. One calls $E$ a *fundamental solution.*

The preceding construction gives a fundamental solution with optimal local regularity properties (cf. Hörmander [1, section 3.1] where references to earlier literature are also given). The construction is clearly applicable without change if $P$ depends on parameters (cf. Trèves [8], [9]). Summing up:

THEOREM 1.1.1. *There exists a continuous map* $E: Pol^0(m) \to \mathscr{D}'(\mathbf{R}^n)$ *such that* $P(D)\, E(P) = \delta$ *for every* $P \in Pol^0(m)$.

## 1.2. *Global existence theorems*

Let $X$ be an open set in $\mathbf{R}^n$ and let $C^\infty(X)$, $\mathscr{D}'(X)$, $\mathscr{D}'^F(X)$ be the set of all infinitely differentiable functions, distributions and distributions of finite order in $X$. We shall consider the equation

$$P(D)u = f \tag{1.2.1}$$

with $u$ and $f$ in one of these spaces. Since $f$ may then be very large at the boundary, conditions have to be imposed on $X$ and on $P$.

THEOREM 1.2.1. *The following four conditions are equivalent:*

(i) *For every* $f \in C^\infty(X)$ *there is a solution* $u \in C^\infty(X)$ *of* (1.2.1).

(ii) *For every* $f \in \mathscr{D}'^F(X)$ *there is a solution* $u \in \mathscr{D}'^F(X)$ *of* (1.2.1).

(iii) *For every* $f \in C^\infty(X)$ *there is a solution* $u \in \mathscr{D}'(X)$ *of* (1.2.1).

(iv) *For every compact set* $K \subset X$ *there is a compact set* $K' \subset X$ *such that*

$$v \in \mathscr{E}'(X)\,, \ \operatorname{supp} P(-D)v \subset K \Rightarrow \operatorname{supp} v \subset K'\,. \tag{1.2.2}$$

The theorem is essentially due to Malgrange [1] (see also Hörmander [1, section 3.5]). Since the proof just consists of abstract functional analysis the equivalence of (i) and (iv) remains valid with minor changes of (iv) if $P$ is a differential operator with variable coefficients for which a semi-global existence theory is established. The operator $P(-D)$ in (1.2.2) should of course be replaced by the formal adjoint ${}^tP$ then. When $f \in \mathscr{D}'(X)$ we have similar results:

THEOREM 1.2.2. *Suppose that* $P(D)$ *defines a surjective map* $\mathscr{D}'(X) \to \to \mathscr{D}'(X)/C^\infty(X)$. *For every compact set* $K \subset X$ *there is then a compact set* $K' \subset X$ *such that*

$$v \in \mathscr{E}'(X)\,, \ \text{sing supp } P(-D)v \subset K \Rightarrow \text{sing supp } v \subset K'\,. \tag{1.2.3}$$

Here sing supp $v$ denotes the smallest closed set such that $v \in C^\infty$ in the complement. Although Theorem 1.2.2 is not formally identical to Theorem 3.6.3 in Hörmander [1], the proof of that theorem is actually a proof of Theorem 1.2.2 above. A similar result is sometimes but not always valid for operators with variable coefficients.

*Example* 1.2.3. For the differential operator $P = \sin \pi x\, d/dx$ on $\mathbf{R}$ we have $P\mathscr{D}'(\mathbf{R}) = \mathscr{D}'(\mathbf{R})$. In fact, to solve the equation $Pu = f$ we have only to solve first a simple division problem and then an ordinary differential equation. However, ${}^tPv = 0$ for all measures $v$ supported by the integers so the analogue of (1.2.3) would be false.

On the other hand, the converse of Theorem 1.2.2 is very general:

THEOREM 1.2.4. *Let $X$ be a $C^\infty$ manifold and $P$ a continuous linear map $\mathscr{D}'(X) \to \mathscr{D}'(X)$ whose restriction to $C^\infty(X)$ is a (continuous) map into $C^\infty(X)$. Denote by ${}^tP$ the adjoint with respect to some positive density in $X$, which is then a continuous operator in $C_0^\infty(X)$ and in $\mathscr{E}'(X)$. Assume that to every compact set $K$ in $X$ there is another compact set $K'$ in $X$, which can be taken empty if $K$ is empty, such that*

$$(1.2.3)' \quad v \in \mathscr{E}'(X), \ \text{sing supp}\ {}^tPv \subset K \Rightarrow \text{sing supp}\ v \subset K'.$$

*Then $P$ defines a surjective map $\mathscr{D}'(X) \to \mathscr{D}'(X)/C^\infty(X)$.*

From Theorems 1.2.1, 1.2.2 and 1.2.4 we obtain

COROLLARY 1.2.5. *If $X$ is an open set in $\mathbf{R}^n$ we have $P(D)\mathscr{D}'(X) = \mathscr{D}'(X)$ if and only if to every compact set $K \subset X$ there is another compact set $K' \subset X$ such that* (1.2.2) *and* (1.2.3) *are valid.*

Corollary 1.2.5 was proved in section 3.6 of Hörmander [1]. A proof of Theorem 1.2.4 is easily extracted from the proof of Theorem 3.6.4 there, but we give it in full here as a typical case of the arguments relating theorems on existence of solutions to theorems on regularity of solutions.

*Proof of Theorem* 1.2.4. It is sufficient to prove that for every $f \in \mathscr{D}'(X)$ there is a continuous semi-norm $q$ on $C_0^\infty(X)$ and a sequence $\psi_j \in C_0^\infty(X)$ with locally finite supports such that

$$(1.2.4) \qquad |f(\varphi)| \leqq q({}^tP\varphi) + \sum |<\varphi, \psi_j>|, \quad \varphi \in C_0^\infty(X).$$

In fact, if we apply the Hahn-Banach theorem to extend the map

$$({}^tP\varphi, <\varphi,\psi_1>, <\varphi,\psi_2>, \ldots) \to f(\varphi)$$

to a linear form on $C_0^\infty(X) \oplus l^1$, we obtain an element $u \in \mathscr{D}'(X)$ and a bounded sequence $a_j$ such that

$$f(\varphi) = u({}^tP\varphi) + \sum a_j <\varphi, \psi_j>, \quad \varphi \in C_0^\infty(X),$$

which means that $f = Pu + \sum a_j \psi_j$. To prove (1.2.4) we first replace $|f(\varphi)|$ by an arbitrary continuous semi-norm $F(\varphi)$ in $C_0^\infty(X)$ which is stronger than the maximum norm for example. We want to prove that

(1.2.4)′ $\quad F(\varphi) \leqq C\big(q({}^tP\varphi)+\sum|<\varphi,\psi_j>|\big)\ ,\ \varphi\in C_0^\infty(X)\,.$

Choose an increasing sequence $K_j$ of compact sets in $X$ with union $X$ and $K_0 = \varnothing$ and choose for every $j$ a corresponding $K_j'$ according to the hypothesis so that $K_0' = \varnothing$ and $K_j'$ is in the interior of $K_{j+1}'$. (Note that we require manifolds to be countable at infinity.)

LEMMA 1.2.6. *Assume that* (1.2.4)′ *is valid when* $\varphi \in C_0^\infty(K_j')$. *If* $\varepsilon > 0$ *one can find another semi-norm* $q'$ *on* $C_0^\infty$ *such that* $q'(\psi) = q(\psi)$ *when* $\psi \in C_0^\infty(K_{j-1})$ *and* (1.2.4)′ *is valid when* $\varphi \in C_0^\infty(K_{j+1}')$ *if* $C$ *is replaced by* $(1+\varepsilon)\,C$, $q$ *is replaced by* $q'$ *and the functions* $\psi_j$ *are supplemented by a finite number of functions in* $C_0^\infty(\complement K_{j-1}')$.

If we note that the hypothesis of the lemma is trivially fulfilled when $j = 0$ and if we iterate this conclusion with a sequence $\varepsilon_j$ with $\Pi(1+\varepsilon_j) < \infty$, we conclude from the lemma that (1.2.4)′ is valid for suitable $C$, $q$ and $\psi_j$.

*Proof of Lemma* 1.2.6. Let $\Phi$ be the completion of $C_0^\infty(K_{j+1}')$ in the weakest topology in which $F(\varphi)$ is continuous and the map from $\varphi$ to the restriction of ${}^tP\varphi$ to $\complement K_{j-1}$ is continuous with values in $C^\infty(\complement K_{j-1})$. Then $\Phi$ is contained in the space of continuous functions with support in $K_{j+1}'$, and for every $\varphi \in \Phi$ we have ${}^tP\varphi \in C^\infty(\complement K_{j-1})$, hence $\varphi \in C^\infty(\complement K_{j-1}')$. It follows that restricting functions in $\Phi$ to $\complement K_{j-1}'$ gives a continuous map from $\Phi$ to $C^\infty(\complement K_{j-1}')$ so bounded sequences in $\Phi$ are also bounded in the latter space.

Let $\chi_1, \chi_2, \ldots$ be a dense sequence in $C_0^\infty(\complement K_{j-1}')$, and let $q_1, q_2, \ldots$ be semi-norms defining the topology in $C^\infty(\complement K_{j-1})$. For convenience we choose these so that $2q_j \leqq q_{j+1}$ for every $j$. Then we claim that for some integer $N$ and all $\varphi \in C_0^\infty(K_{j+1}')$

(1.2.5)
$$F(\varphi) \leqq C(1+\varepsilon)\big(q({}^tP\varphi)+\sum|<\varphi,\psi_j>|+q_N({}^tP\varphi)+N\sum_{k<N}|<\varphi,\chi_k>|\big)\,.$$

This would prove the lemma. Now if (1.2.5) is not valid for any $N$ we can choose a sequence $\varphi_N \in C_0^\infty(K_{j+1}')$ such that

$$F(\varphi_N) = C(1+\varepsilon)\,,\ q({}^tP\varphi_N) + \sum|<\varphi_N,\psi_j>| \leqq 1$$

and ${}^tP\varphi_N \to 0$ in $C^\infty(\complement K_{j-1})$, $<\varphi_N, \chi_k> \to 0$ for every $k$ as $N \to \infty$. But then $\varphi_N$ is relatively compact in $C^\infty(\complement K_{j-1}')$ and every limit is orthogonal to all $\chi_k$ and therefore equal to 0. Thus $\varphi_N \to 0$ in $C^\infty(\complement K'_{j-1})$. Choose now a function $\psi \in C_0^\infty(K'_j)$ which is 1 in a neighborhood of $K'_{j-1}$. Then it follows that $(1-\psi)\varphi_N \to 0$ in $C_0^\infty$. If $\varphi'_N = \psi\varphi_N$ we obtain for large $N$

$$F(\varphi'_N) > C(1+2\varepsilon/3)\,,\ q({}^tP\varphi'_N) + \sum_j | < \varphi'_N, \psi_j > | < 1 + \varepsilon/3\,.$$

Since $\varphi'_N \in C_0^\infty(K'_j)$, this contradicts the hypothesis that (1.2.4) is valid for such functions. The proof is complete.

It is a simple exercise in Fredholm theory to show that the hypotheses of Theorem 1.2.4 imply that for every compact set $K \subset X$ the space $N(K)$ of all $v \in \mathscr{E}'(K)$ with ${}^tPv = 0$ is finite dimensional, and that the equation $Pu = f \in \mathscr{D}'(X)$ can be fulfilled on a neighborhood of $K$ with $u \in \mathscr{D}'(X)$ if (and only if) $f$ is orthogonal to $N(K)$. In fact, for this we only need that sing supp ${}^tPv = \varnothing$ implies sing supp $v = \varnothing$ when $v \in \mathscr{E}'(X)$. Thus results on the regularity of solutions of differential equations imply theorems on the existence of solutions, and for this reason we shall mainly pay attention to the regularity of solutions in these lectures.

Returning to differential operators with constant coefficients we introduce a slight modification of the terminology in Hörmander [1].

*Definition* 1.2.7. The open set $X$ in $\mathbf{R}^n$ is called $P$-convex with respect to supports (resp. singular supports) if for every compact set $K \subset X$ there is another compact set $K' \subset X$ such that (1.2.2) (resp. (1.2.3)) is valid.

The use of the term " convex " will be justified by the discussion of the geometric meaning in sections 1.3 and 1.4. Here we just note that convex sets are $P$-convex both with respect to supports and singular supports. An elementary argument using the translation invariance of $P(-D)$ also gives (see Theorem 3.5.2 in Hörmander [1]):

THEOREM 1.2.8. *Let $x \to |x|$ denote any norm in $\mathbf{R}^n$ and set for closed sets $F$ in $X$*

$$d(F, \complement X) = \inf_{x\in F, y\notin X} |x - y|\,.$$

*Then $X$ is $P$-convex with respect to supports if and only if*

$$d(\operatorname{supp} P(-D)v, \complement X) = d(\operatorname{supp} v, \complement X), v \in \mathscr{E}'(X)\,, \tag{1.2.6}$$

*and with respect to singular supports if and only if*

$$d(\text{sing supp } P(-D)v, \complement X) = d(\text{sing supp } v, \complement X)\,, v \in \mathscr{E}'(X)\,. \tag{1.2.7}$$

The analogy of the notions of $P$-convexity in Definition 1.2.7 to holomorphic convexity in the theory of functions of several complex variables is obvious. The purpose of the next two sections is to discuss some analogues of pseudo-convexity.

### 1.3. *Geometric conditions for P-convexity with respect to supports*

Throughout this section we denote by $X$ an open set in $\mathbf{R}^n$ and by $P(D)$ a partial differential operator with constant coefficients. The following two simple theorems describe the conditions for $P$-convexity of $X$ which involve only $P$ or only $X$.

THEOREM 1.3.1. *$X$ is $P$-convex with respect to supports for every $P$ if and only if every component of $X$ is convex in the usual sense.*

THEOREM 1.3.2. *Every $X$ is $P$-convex with respect to supports if and only if $P$ is elliptic.*

Ellipticity means, if $P$ is of degree $m$ and

$$P(\xi) = P_m(\xi) + P_{m-1}(\xi) + \ldots$$

is the decomposition of $P$ in a sum of homogeneous terms $P_j$ of degree $j$, that

$$P_m(\xi) \neq 0 \quad if \quad 0 \neq \xi \in \mathbf{R}^n . \tag{1.3.1}$$

$P_m$ is called the principal part of $P$. Solutions of the equation $P_m(\xi) = 0$ with $\xi \neq 0$ (and $\xi \in \mathbf{R}^n$) are called (real) *characteristics*. A hypersurface is said to be characteristic when the normal is characteristic. A characteristic point $\xi$ with $dP_m(\xi) \neq 0$ is said to be simply characteristic, and the projection in $\mathbf{R}^n$ of a complex line in $\mathbf{C}^n$ with direction $(\partial P_m/\partial\xi_1, \ldots, \partial P_m/\partial\xi_n)$ will then be called a *bicharacteristic* corresponding to $\xi$. It may be of dimension 1 or 2.

Now observe that $X$ is not $P$-convex with respect to supports if for some open set $Y \subset\subset X$ (i.e. $Y$ is relatively compact in $X$) there is a distribution $u \in \mathcal{D}'(Y)$ with

(1.3.2)
$$d(\operatorname{supp} u, \complement X) < \min\left(d(\partial Y \cap \overline{\operatorname{supp} u}, \complement X), d(\operatorname{supp} P(-D)\,u, \complement X)\right).$$

In fact, (1.2.6) is not valid if $v = \varphi u$ and $\varphi \in C_0^\infty(Y)$ is equal to 1 in a sufficiently large compact subset of $Y$. If $P(-D)\,u = 0$ in $Y$ and $u = 0$ at a part of the boundary this leads to necessary conditions for $P$-convexity. In particular one can use the fact that there is a solution of $P(-D)\,u = 0$ with support equal to any half space with characteristic boundary (Hörmander [1, Theorem 5.2.2]). In stating the result we shall say that a function $f$ in $X$ satisfies the minimum principle in a closed set $F$ if for every compact set $K \subset F \cap X$ we have

$$\min_{x \in K} f(x) = \min_{x \in \partial_F K} f(x)$$

where $\partial_F K$ is the boundary of $K$ as a subset of $F$. We write $d_X(x) = d(\{x\}, \complement X)$.

THEOREM 1.3.3. *If $X$ is $P$-convex with respect to supports, then $d_X(x)$ satisfies the minimum principle in any characteristic hyperplane. When $n = 2$ this means that every component of $X$ is convex in the direction of any characteristic line, and this condition is also sufficient for $X$ to be $P$-convex with respect to supports.*

For the proof we refer to section 3.7 in Hörmander [1], where it is also shown that Theorem 1.3.3 implies the necessity in Theorems 1.3.1 and 1.3.2. When $n > 2$ the necessary condition in Theorem 1.3.3 is far from sufficient, however, for there are many characteristic surfaces which are not planes and a classical theorem of Goursat allows one to construct local solutions vanishing on one side of any simply characteristic surface. Thus Malgrange [2] proved (see also Theorem 3.7.3 in Hörmander [1]):

THEOREM 1.3.4. *Let $P(D)$ be a differential operator such that the principal part $P_m(D)$ has real coefficients and let $X$ be $P$-convex with respect to supports. At every simply characteristic $C^2$ boundary point the normal curvature of $\partial X$ in the direction of the corresponding bicharacteristic must then be non-negative.*

Actually the proof of Malgrange gives somewhat more, namely that for no boundary point $x_0$ with simply characteristic normal $N_0$ does there exist a cylinder with $C^2$ boundary and the corresponding bicharacteristic as generator containing $x_0$ and contained in $X \cup \{x_0\}$ near $x_0$. This improvement is given in a different form in Trèves [1].

In the proof of Theorem 1.3.4 a simply characteristic surface is constructed by means of the Hamilton-Jacobi integration theory. Using this theory for the system of equations Re $P_m$ (grad $\varphi$) $=$ 0, Im $P_m$ (grad $\varphi$) $=$ 0 (see e.g. Carathéodory [1, Chapter IV]) one obtains

THEOREM 1.3.5. *Let $X$ be $P$-convex with respect to supports and have a $C^2$ boundary. At every boundary point where the normal is simply characteristic and the corresponding bicharacteristic is two dimensional the normal curvature of $\partial X$ in some direction in the bicharacteristic must then be non-negative.*

So far we have only given necessary conditions for $P$-convexity. To give sufficient conditions means to prove uniqueness theorems. For example, the sufficiency in Theorem 1.3.2 follows from the fact that solutions of homogeneous elliptic equations are real analytic and therefore have a property of unique continuation. In general we have available the uniqueness theorem of Holmgren (see Hörmander [1, section 5.3]) and variations of it for continuation across characteristic surfaces. (See Trèves [1], Zachmanoglou [1], Bony [1], Hörmander [11, 12].) From the results of Hörmander [11] we obtain, for example, the following theorem which should be compared with Theorem 1.3.4; it is clear that an analogous result can be proved corresponding to Theorem 1.3.5.

THEOREM 1.3.6. *Let $P(D)$ be a differential operator such that the principal part $P_m(D)$ has real coefficients, and let $X$ be an open set in $\mathbf{R}^n$ with a $C^1$ boundary. Then $X$ is $P$-convex with respect to supports if every characteristic boundary point $x_0$ is simple and for every closed interval $I$ on the corresponding bicharacteristic with $x_0 \in I \subset \overline{X}$ at least one end point belongs to $\partial X$.*

The proof of Theorem 3.7.3 in Hörmander [1] gives the following partial converse of Theorem 1.3.5 involving weaker conditions on $P$ and stronger conditions on $X$:

THEOREM 1.3.7. *Let $X$ have a $C^2$ boundary for which all characteristic points with respect to $P$ are simple. Assume that at every characteristic boundary point the normal curvature of $\partial X$ in some direction in the corresponding bicharacteristic is positive. Then it follows that $X$ is $P$-convex with respect to supports.*

For later reference we give a simple modification of Theorem 1.3.2:

THEOREM 1.3.8. *Let $P(D)$ be a differential operator in $\mathbf{R}^n$ which acts along a linear subspace $V$ and is elliptic as an operator in $V$. Then an open set $X$ in $\mathbf{R}^n$ is P-convex with respect to supports if and only if $d_X(x)$ satisfies the minimum principle in any affine space parallel to $V$.*

This ends our quite fragmentary list of results. It is clear that $P$-convexity with respect to supports is insufficiently understood as yet. Further study should lead to improved uniqueness theorems.

## 1.4. *Geometric conditions for P-convexity with respect to singular supports*

As in section 1.3 we denote throughout by $X$ an open set in $\mathbf{R}^n$ and by $P(D)$ a partial differential operator with constant coefficients. Again we start by describing the convexity conditions which only involve $P$ or $X$.

THEOREM 1.4.1. *$X$ is P-convex with respect to singular supports for every $P$ if and only if every component of $X$ is convex in the usual sense.*

THEOREM 1.4.2. *Every $X$ is P-convex with respect to singular supports if and only if $P$ is hypoelliptic.*

Hypoellipticity means that for every distribution $u$

$$\text{sing supp}\, u = \text{sing supp}\, Pu \tag{1.4.1}$$

or equivalently that (Hörmander [1, section 4.1])

$$P^{(\alpha)}(\xi)/P(\xi) \to 0 \text{ when } \xi \to \infty \text{ in } \mathbf{R}^n \text{ if } |\alpha| \neq 0\,. \tag{1.4.2}$$

The sufficiency is well known (see section 3.7 in Hörmander [1]) in Theorem 1.4.1 and is trivial in Theorem 1.4.2. The necessity will follow from more precise results below.

Necessary conditions for $P$-convexity with respect to singular supports can be obtained by noting that $X$ is not $P$-convex in this sense if (1.3.2) is valid for some $u$ and $Y \subset\subset X$ with supports replaced by singular supports. To use this remark we need to know solutions of the equation $P(-D)\,u = 0$ with small singular support. Starting from earlier constructions by Zerner [1] and Hörmander [1, section 8.8] rather general results of this type were proved in Hörmander [7]. A heuristic motivation for these is obtained by noting that for functions represented as Fourier integrals it is the high frequency components that may give rise to singularities. It is therefore natural to consider solutions of the equation $P(D)u = 0$ of the form $u(x) = e^{i<x,\xi>} v(x)$ where $\xi$ is large and a major part of $v$ is composed of exponentials with much smaller frequencies. We have

$$P(D)(e^{i<x,\xi>}v) = e^{i<x,\xi>}P_\xi(D)v$$

where $P_\xi(D) = P(D+\xi)$. With $\tilde{P}(\xi)$ defined by (1.1.7) the normalized polynomials $P_\xi/\tilde{P}(\xi)$ belong to the unit sphere in $Pol(m)$, if $m$ is the degree of $P$. Denote the set of limit points when $\xi \to \infty$ by $L(P)$. It is then natural to expect connections between singular supports of solutions of the equation $P(D)u = 0$ and supports of solutions of $Q(D)u = 0$, $Q \in L(P)$.

*Example* 1.4.3. $P$ is hypoelliptic, that is, $P$ satisfies (1.4.2), if and only if all elements of $L(P)$ are constants (of modulus one).

*Example* 1.4.4. If $\eta$ is a simple characteristic of $P$, then the limits of $P_\xi/\tilde{P}(\xi)$ as $\xi \to \infty$ and $\xi/|\xi| \to \eta/|\eta|$ are of the form

$$a \sum_1^n P_m^{(j)}(\eta) D_j + b$$

where $a \geqq 0$ and $|a|^2 \Sigma |P_m^{(j)}(\eta)|^2 + |b|^2 = 1$. Thus we have a first order operator acting along the bicharacteristic corresponding to $\eta$.

The preceding example suggests an extension of the notion of bicharacteristic. If $Q$ is a polynomial, we write

$$\Lambda(Q) = \{\eta \in \mathbf{R}^n; Q(\xi + t\eta) \equiv Q(\xi)\}$$

for the largest vector space in $\mathbf{R}^n$ along which $Q$ is constant, and we introduce the annihilator

$$\Lambda'(Q) = \{x \in \mathbf{R}^n; <x, \eta> = 0, \ \eta \in \Lambda(Q)\},$$

which is the smallest subspace such that $Q(D)$ operates along $\Lambda'(Q)$. This means that $Q(D)u(0)$ is determined by the restriction of $u$ to $\Lambda'(Q)$ and that $\Lambda'(Q)$ is the smallest subspace of $\mathbf{R}^n$ with this property. When $Q \in L(P)$ is not constant so that $dim\, \Lambda'(Q) > 0$, the planes parallel to $\Lambda'(Q)$ will be called *bicharacteristic spaces* for $P$. (These are the same for $P(D)$ and the adjoint $P(-D)$.) For every such plane $B$ the equation $Q(D)u = 0$ obviously has solutions with supp $u = B$. Arguing along the lines familiar in geometrical optics one can make the heuristic arguments above precise and show that the equation $P(D)u = 0$ has a solution with sing supp $u = B$. This leads to

THEOREM 1.4.5. *If $X$ is $P$-convex with respect to singular supports, it follows that the minimum principle is valid for $d_X$ on all bicharacteristic spaces for $P$.*

When some $Q \in L(P)$ is non-elliptic as an operator in $\Lambda'(Q)$, this result can be improved (see e.g. Corollary 3.5 in Hörmander [7]). However, Theorem 1.4.6 below indicates that it may well be that the condition in Theorem 1.4.5 is sufficient if all $Q \in L(P)$ are elliptic. In this situation we see from Theorem 1.3.8 that the necessary condition in Theorem 1.4.5 means that $X$ is $Q$-convex with respect to supports for every $Q \in L(P)$. It may perhaps be true in more general circumstances that $P$-convexity with respect to singular supports is equivalent to $Q$-convexity with respect to supports for all $Q \in L(P)$.

THEOREM 1.4.6. *$X$ is $P$-convex with respect to singular supports if either of the following conditions is fulfilled:*

i) *$X \cap V$ is convex if $V$ is any bicharacteristic space for $P$;*

ii) *All bicharacteristic spaces are 1-dimensional and $d_X$ satisfies the minimum principle in all of them;*

iii) *All $Q \in L(P)$ are of order $\leq 1$ and $d_X$ satisfies the minimum principle in all bicharacteristic spaces.*

For the cases i) and ii) proofs are given in Hörmander [7]. They depend on modifications of the construction of fundamental solutions given in section 1.1 above. The proof of iii) will be given in section 1.5.

## 1.5. *Propagation of singularities for solutions of operators with first order localizations at infinity*

Let $P(D)$ be a differential operator such that every $Q \in L(P)$ is a first order operator. Since $P(D+\xi) = \Sigma P^{(\alpha)}(\xi) D^\alpha/\alpha\,!$ this means that we assume

$$P^{(\alpha)}(\xi)/\tilde{P}(\xi) \to 0 \quad \text{when} \quad \xi \to \infty \quad \text{if} \quad |\alpha| > 1 . \tag{1.5.1}$$

This condition is analogous to the condition (1.4.2) for hypoellipticity, and it is fulfilled by any product of one hypoelliptic operator and one operator with simple characteristics. If $x \in \mathbf{R}^n$ we denote by $B_x$ the closure of the set of bicharacteristic spaces for $P$ containing $x$. Condition iii) in Theorem 1.4.6 clearly does not change if in addition to bicharacteristic spaces we consider limits of such spaces. (It may be appropriate to call such limits also bicharacteristic.) The last part of Theorem 1.4.6 is therefore a consequence of

THEOREM 1.5.1. *Let* $u \in \mathscr{D}'(X)$ *where* $X \subset \mathbf{R}^n$ *is an open set, and assume that* $P(D)u \in C^\infty(X)$. *If* $x \in \operatorname{sing\,supp} u$ *it follows that for some* $b \in B_x$ *the component of* $X \cap b$ *containing* $x$ *is a subset of* sing supp $u$.

With $X_0 = X \setminus \operatorname{sing\,supp} u$ there is an equivalent statement which is more convenient in the proof:

THEOREM 1.5.2. *Let* $X_0 \subset X$ *be open,* $u \in \mathscr{D}'(X)$, $P(D)u \in C^\infty(X)$ *and* $u \in C^\infty(X_0)$. *If* $x \in X$ *and the component of* $X \cap b$ *containing* $x$ *meets* $X_0$ *for every* $b \in B_x$, *it follows that* $u \in C^\infty$ *in a neighborhood of* $x$.

Since $B_x$ is compact the hypothesis will still be fulfilled if $X$ is replaced by a sufficiently large relatively compact subset. We may then assume without restriction that $u \in \mathscr{E}'(\mathbf{R}^n)$.

The first step in the proof is to localize the spectrum of $u$. Let $\rho$ be any number with $0 < \rho < 1$. As in Hörmander [7] we can choose a partition of unity $1 = \Sigma_0^\infty \psi_j$ in $\mathbf{R}^n$ such that

i) $$0 \leqq \psi_j \in C_0^\infty$$

and

$$|\xi - \xi_j| < C|\xi_j|^\rho \ \textit{if} \ \xi \in \operatorname{supp} \psi_j; \ \psi_j(\xi) = 1 \ \textit{if} \ |\xi - \xi_j| < c|\xi_j|^\rho,$$

for some constants $c$, $C$ and a sequence $\xi_j \in \mathbf{R}^n$.

ii) $$\sup |D^\alpha \psi_j| \leqq C_\alpha |\xi_j|^{-\rho|\alpha|}.$$

Note that i) implies that

$$\sum_0^\infty \xi_j^{n\rho - a} \leqq C \int_{|\xi|>1} |\xi|^{-a} d\xi < \infty \ \textit{if} \ a > n.$$

Condition ii) implies that for every positive integer $N$

(1.5.2) $$|\mathscr{F}^{-1}\psi_j(x)| < C_N |\xi_j|^{n\rho} (1 + |x||\xi_j|^\rho)^{-N}.$$

LEMMA 1.5.3. *If $u \in \mathscr{E}'(\mathbf{R}^n)$ is of order $\mu$ and $\hat{u}_j = \psi_j \hat{u}$, then*

$$\sup |u_j| \leqq C |\xi_j|^{\mu+n\rho}. \tag{1.5.3}$$

*For an open set $Y$ we have $u \in C^\infty(Y)$ if and only if for every compact set $K \subset Y$ and every positive integer $N$*

$$\sup_{x \in K} |u_j(x)| < C_{N,K} |\xi_j|^{-N}. \tag{1.5.4}$$

*Proof.* (1.5.3) is obvious and so is (1.5.4) for every $K$ if $u \in C_0^\infty(\mathbf{R}^n)$. In view of (1.5.2) it is also clear that (1.5.4) is valid outside supp $u$. Combination of these facts proves that (1.5.4) is valid if $K$ does not meet sing supp $u$. On the other hand, assume that (1.5.4) is valid in a neighborhood of $K$. Since $u$ is of exponential type at most $C|\xi_j|$ it follows from (1.5.3) that

$$|u_j(z)| \leqq C |\xi_j|^{\mu+n\rho} \exp(C|\xi_j||\operatorname{Im} z|), \quad z \in \mathbf{C}^n.$$

Hence $|u_j(z)| \leqq C|\xi_j|^{\mu+n\rho}$ when $|\operatorname{Im} z| < 1/|\xi_j|$. Using for example the three lines theorem (cf. John [1]) we conclude that $u_j(z) = 0(|\xi_j|^{-N})$ for every $N$ in the set of points in $\mathbf{C}^n$ at distance at most $1/2n|\xi_j|$ from $K$. But then Cauchy's inequality shows that $D^\alpha u_j(x) = 0(|\xi_j|^{-N})$ for all $\alpha$ and $N$ when $x \in K$, which proves that $\Sigma D^\alpha u_j(x)$ is uniformly convergent in $K$ for every $\alpha$. Hence $u \in C^\infty$ in the interior of $K$ which proves the lemma.

We shall apply Lemma 1.5.3 to the distributions $u$ and $f = P(D)u$ which occur in Theorem 1.5.2. Thus we define $u_j$ and $f_j$ by $\hat{u}_j = \psi_j \hat{u}$ and $\hat{f}_j = \psi_j \hat{f}$. Then we have (1.5.4) for compact subsets of $X_0$, and if $u$ is replaced by $f$ we have (1.5.4) for compact subsets of $X$. The equation $P(D)u = f$ implies that $P(D)u_j = f_j$.

The spectrum of $u_j$ is concentrated near $\xi_j$ so we introduce

$$v_j(x) = u_j(x)e^{-i<x,\xi_j>}, \quad g_j(x) = f_j(x)e^{-i<x,\xi_j>}/\tilde{P}(\xi_j).$$

The equation $P(D)u_j = f_j$ then becomes

$$\tilde{P}(\xi_j)^{-1} P_{\xi_j}(D) v_j = g_j. \tag{1.5.5}$$

Here $v_j$ and $g_j$ have the properties stated above for $u_j$ and $f_j$, and they are of exponential type $C|\xi_j|^\rho$ by the property i) of the partition of unity.

By Proposition 2.4 in Hörmander [7] we can for every $j$ choose $Q_j \in L(P)$ so close to $P_{\xi_j}/\tilde{P}(\xi_j)$ that $P_{\xi_j}(D)/\tilde{P}(\xi_j) = Q_j(D) - R_j(D)$ where

$$\tilde{R}_j(0) \leqq C|\xi_j|^{-b} \tag{1.5.6}$$

for some $b > 0$. We rewrite (1.5.5) in the form

$$(1.5.5)' \qquad (Q_j(D) - R_j(D))\, v_j = g_j .$$

To take advantage of the fact that the coefficients of $R_j$ are small we multiply both sides by $Q_j(D)^{k-1} + Q_j(D)^{k-2} R_j(D) + ... + R_j(D)^{k-1}$ and obtain

$$(1.5.7) \quad Q_j(D)^k v_j = R_j(D)^k v_j + \sum_{\nu=1}^{k} Q_j(D)^{k-\nu} R_j(D)^{\nu-1} g_j .$$

The terms in the sum are $O(|\xi_j|^{-N})$ for all $N$ on compact subsets of $X$. Since $v_j$ satisfies (1.5.3) and is of exponential type $C\,|\,\xi_j\,|^\rho$, we have for every $\alpha$ by Bernstein's inequality

$$|\,D^\alpha v_j\,| < C_\alpha\,|\,\xi_j\,|^{a+|\alpha|\rho}$$

where we have written $a = \mu + n\rho$. Using (1.5.6) we therefore obtain

$$|\,R_j(D)^k v_j\,| < C_k\,|\,\xi_j\,|^{a+k(m\rho-b)} .$$

If we choose $\rho$ so small that $m\rho < b$, the right hand side will decrease like any desired power of $1/|\,\xi_j\,|$ if $k$ is large. To complete the proof of the theorem it is therefore sufficient to show that for solutions of an equation $Q(D)^k v = h$ where $Q \in L(P)$, $h$ is small in $X$, $v$ is bounded in $X$ and small in $X_0$, it is true uniformly with respect to $k$ and $Q$ that $v$ is small near the point $x$ in Theorem 1.5.2. This is essentially a consequence of classical convexity theorems but the uniformity needed here forces us to reconsider these carefully.

1) Let $I \subset \mathbf{R}$ be an interval with 0 in its interior and let $I_0$ be another interval of positive length $\subset I$. Then there exist constants $C$ and $\delta$, $0 < \delta < 1$, such that

$$(1.5.8) \quad |\,u(0)\,| \leqq C^k (\sup_{I_0} |u|)^\delta \, (\sup_I |u|)^{1-\delta} \;\; if \;\; (d/dx - \lambda)^k u = 0 .$$

Here $C$ and $\delta$ depend on $I$ and $I_0$ but are independent of $k$ and the complex number $\lambda$. To prove (1.5.8) we note that $u(x) = e^{\lambda x} p(x)$ where $p$ is a polynomial of degree $k - 1$. Assuming for example that $\operatorname{Re} \lambda \geqq 0$ we choose a closed interval $I_1 \subset I$ in the open positive $x$-axis. For suitable positive constants

$$\sup_{I_0} |\,u\,| \geqq e^{-c_0 \operatorname{Re}\lambda} \sup_{I_0} |\,p\,| , \quad \sup_{I} |\,u\,| \geqq e^{c_1 \operatorname{Re}\lambda} \sup_{I_1} |\,p\,| .$$

By classical inequalities of Tschebyscheff we have for some constant $C$

$$|p(0)| \leqq C^k \sup_{I_0} |p|, \quad |p(0)| \leqq C^k \sup_{I_1} |p|.$$

Hence we obtain (1.5.8) if $\delta c_0 \leqq (1-\delta)\, c_1$, that is, if $\delta \leqq c_1/(c_0+c_1)$.

2) Let $X_0 \subset X_1$ be open sets in **C** such that some point of $X_0$ is in the component of 0 in $X_1$. Then one can find compact sets $K_j \subset X_j$ and constants $C$, $\delta$ with $0 < \delta < 1$ such that

$$(1.5.9) \quad |u(0)| \leqq C^k (\sup_{K_0} |u|)^\delta (\sup_{K_1} |u|)^{1-\delta} \; if \; (\partial/\partial\bar{z} - \lambda)^k u = 0.$$

Here $C$ is independent of $k$ and of $\lambda$. A substitution $u = ve^{i<x,\xi>}$ where $(i\xi_1 - \xi_2)/2 = \lambda$ and $\xi$ is real reduces the proof to the case $\lambda = 0$. It is sufficient to prove that if $0 < r < r_1$, $0 < r_0 < r_1$ then

$$(1.5.10) \quad \sup_{|z|<r} |u(z)| \leqq C^k (\sup_{|z|<r_0} |u|)^\delta (\sup_{|z|<r_1} |u|)^{1-\delta} \text{ if } (\partial/\partial\bar{z})^k u = 0$$
$$\text{when } |z| < r_1,$$

for if we join 0 to a point in $X_0$ by a polygon, repeated use of (1.5.10) will yield (1.5.9). For $k = 1$ the inequality (1.5.10) is included in the three circles theorem of Hadamard. In the general case we note that

$$u(z) = \sum_0^{k-1} \bar{z}^j u_j(z)$$

where $u_j$ is analytic. When $|z| = R < r_1$ we have $\bar{z} = R^2/z$ and therefore

$$\left|\sum_0^{k-1} R^{2j} z^{k-1-j} u_j(z)\right| \leqq r_1^{k-1} \sup_{|z|<r_1} |u(z)| \text{ when } |z| \leqq R < r_1.$$

If $|z| \leqq r_1' < r_1$ and $R$ varies between $r_1'$ and $r_1$ it follows from the classical estimates of Tschebyscheff for the coefficients of a polynomial (in $R$) that

$$\sup_{|z|<r_1'} |u_j(z)| \leqq C^k \sup_{|z|<r_1} |u(z)|.$$

A similar estimate is valid if we replace $r_1$ by $r_0$ and $r_1'$ by a positive number $r_0' < r_0$, But this reduces the proof of (1.5.10) to the case $k = 1$ where as already pointed out the inequality follows from the three circles theorem of Hadamard.

We can now prove the main lemma. Let $M$ be a family of first order differential operators $Q(D)$ with $\tilde{Q}(0) = 1$. Assume that $Q \in M$ implies $Q_\eta/\tilde{Q}(\eta) \in M$ for $\eta \in \mathbf{R}^n$ and that $M$ is closed in *Pol* (1). Denote by $B$ the closure of the set of all $\Lambda'(Q)$ with $Q \in M$.

LEMMA 1.5.4. *Assume that $X_0 \subset X$ are open sets in $\mathbf{R}^n$ with $0 \in X$ and assume that for every $b \in B$ the component of 0 in $b \cap X$ contains some point in $X_0$. Then one can find compact sets $K_0 \subset X_0$ and $K \subset X$ such that*

$$(1.5.11) \quad |u(0)| \leqq C^k (\sup_{K_0} |u| + N_k(u))^\delta (\sup_K |u| + N_k(u))^{1-\delta},$$

$$N_k(u) = \sum_{|\alpha| \leqq k+n+1} \sup_K k^{k-|\alpha|} |D^\alpha Q(D)^k u|,$$

*if $u \in C^\infty(X)$, $Q \in M$, and $k$ is a positive integer. The constants $C$ and $\delta$ do not depend on $u$, $Q$ or $k$.*

*Proof.* We shall first verify (1.5.11) when $Q(D)^k u = 0$ in a neighborhood of a sufficiently large compact set $K \subset X$. When $Q(D)$ is any fixed first order operator with $\Lambda'(Q) \in B$ this case of (1.5.11) is contained in (1.5.8) and (1.5.9). When $dim\,\Lambda'(Q) = 1$ the same constants and compact sets can be used for all $Q$ with $\Lambda'(Q)$ close to a fixed line in $B$ so the compactness of $S^{n-1}$ shows that we can use the same constant for all $Q \in M$ with $dim\,\Lambda'(Q) = 1$. When $dim\,\Lambda'(Q) = 2$ we first note as in the proof of (1.5.9) that $Q$ may be replaced by a real translate which contains no term of order 0. Let $M_0 \subset M$ be the closure of the set of all $Q \in M$ with $dim\,\Lambda'(Q) = 2$ and $Q(0) = 0$, $\tilde{Q}(0) = 1$. It follows from (1.5.9) that (1.5.11) is valid when $Q(D)^k u = 0$ on a large compact subset of $X$, uniformly for all $Q \in M_0$ in a neighborhood of an element with $dim\,\Lambda'(Q) = 2$. The operators in $M_0$ near an element $Q_0$ with $dim\,\Lambda'(Q_0) = 1$ can after multiplication by a factor of modulus 1 be written

$$Q(D)u = \langle a, \operatorname{grad} u \rangle + i \langle b, \operatorname{grad} u \rangle$$

where $a$ and $b$ are real, $a$ is orthogonal to $b$, $|a|^2 + |b|^2 = 1$ and $a$ is close to a unit vector in $\Lambda'(Q_0)$. Introducing $a$ and $b$ as basis vectors in $\Lambda'(Q)$ we obtain the homogeneous case of (1.5.11) from (1.5.9) with constants and compact sets depending only on $Q_0$.

It remains to extend (1.5.11) to the inhomogeneous case. Let $f \in C_0^\infty(K_1)$ where $K_1 \subset\subset X$ is a neighborhood of the compact set $K$ obtained in the proof for the homogeneous case. We wish to solve the equation

$$(1.5.12) \qquad Q(D)^k u = f$$

when $Q \in M$. Since $\tilde{Q}(0) = 1$ and $1 = \tilde{Q}(0) \leqq (1+|\xi|)\,\tilde{Q}(\xi)$ we have $\tilde{Q}(\xi) \geqq (1+|\xi|)^{-1}$. With the notations of (1.1.6) it follows that

$$|Q(\xi+\zeta)| \geqq C(1+|\xi|)^{-1} \quad \text{if} \quad \Phi(Q_\xi, \zeta) \neq 0.$$

Hence the solution of (1.5.12) given by

$$u(x) = (2\pi)^{-n} \int d\xi \int e^{i<x,\xi+\zeta>} \hat{f}(\xi+\zeta)\, Q(\xi+\zeta)^{-k}\, \Phi(Q_\xi, \zeta)\, d\lambda(\zeta)$$

has on every compact set an estimate of the form

$$(1.5.13) \qquad |u(x)| \leqq C^k \sum_{|\alpha| \leqq k+n+1} \sup |D^\alpha f| .$$

Here we have of course used the elementary and familiar fact that the right hand side of (1.5.13) bounds $(1+|\xi|)^{k+n+1} |\hat{f}(\xi)|$.

To prove (1.5.11) we just choose a function $\chi \in C_0^\infty(K_1)$ with $\chi = 1$ near $K$, $|D^\alpha \chi| \leqq (Ck)^{|\alpha|}$, $|\alpha| \leqq k + n + 1$ (see e.g. Hörmander [11]), and solve as just explained the equation

$$Q(D)^k u_0 = f = \chi Q(D)^k u .$$

(Since we only need to know that (1.5.11) is valid for some constant depending on $k$ instead of $C^k$ it would be sufficient to use any fixed $\chi$.) For $u_0$ we have the bound (1.5.13), and the estimate (1.5.11) is valid with $u$ replaced by $u_1 = u - u_0$. Summing up, we obtain (1.5.11) with $K_1$ instead of $K$.

*End of proof of Theorem* 1.5.2. We may assume that the point $x$ in the theorem is the origin. Then the hypotheses of Lemma 1.5.4 are fulfilled with $M = L(P)$. In view of the translation invariance of (1.5.11) it follows that if $V$ is a compact connected neighborhood of 0 such that $K_0 + V$ and $K + V$ are contained in $X_0$ and $X$ respectively, then

$$(1.5.11)' \quad \sup_V |v| \leqq C^k (\sup_{K_0+V} |v| + N)^\delta (\sup_{K+V} |v| + N)^{1-\delta}, \quad v \in C^\infty, Q \in L(P),$$

where we have written

$$N = \sum_{|\alpha| \leqq k+n+1} \sup_{K+V} k^{k-|\alpha|} |D^\alpha Q(D)^k v| .$$

We shall apply this estimate with $v = v_j$ and $Q = Q_j$ using (1.5.7). We recall that $v_j = 0(|\xi_j|^{-N})$ in $K_0 + V$ for every $N$ and that a similar estimate is valid in $K + V$ for any derivative of the sum in (1.5.7). Furthermore, since $R_j(D)^k v_j$ is of exponential type $C|\xi_j|^\rho$ we obtain

$$\sum_{|\alpha| \leqq k+n+1} \sup_{K+V} |D^\alpha R_j(D)^k v_j| \leqq C_k |\xi_j|^{a_1+k((m+1)\rho-b)}$$

where $a_1 = a + (n+1)\rho$. We choose $\rho$ so small that $(m+1)\rho - b < -b/2$. Then (1.5.11)′ gives for large enough $k$

$$\sup_V |v_j| \leqq C_k (|\xi_j|^{-kb/2})^\delta (|\xi_j|^{\mu+n\rho})^{1-\delta} .$$

Since $\delta > 0$ is independent of $k$ we obtain by choosing $k$ large that $v_j = O(|\xi_j|^{-N})$ on $V$ for all $N$. In view of Lemma 1.5.3 it follows that $v \in C^\infty$ in a neighborhood of 0, which completes the proof of Theorem 1.5.2.

*Remark.* The importance of "Hölder estimates" for the study of propagation of singularities has been emphasized by John [1]. He proved results of the form (1.5.11) for a *fixed* $Q$ which is elliptic as an operator in $\Lambda'(Q)$. However, no study has yet been made of the required uniformity in $Q \in L(P)$ for higher order elliptic operators $Q$.

A number of special cases of Theorem 1.5.1 occur in the literature; see Hörmander [1, section 8.8], Grušin [1], Hörmander [7]. The corresponding question has also been much studied for variable coefficients (see Chapter III) and so has the analogous question with $C^\infty$ replaced by real analytic functions (and sometimes distributions replaced by hyperfunctions); see Andersson [1], Kawai [1], [2], Hörmander [11].

## 1.6. *General wave front sets*

Additional information can be obtained from the proof of Theorem 1.5.2 if one considers not only where in $X$ that the sequence $v_j$ is not $O(|\xi_j|^{-N})$ for all $N$ as $j \to \infty$ but also for which subsequences of $\{\xi_j\}$ that this occurs. We shall now introduce some concepts which allow us to state such conclusions. The simplest and perhaps most natural one is the compactification of $\mathbf{R}^n$ by a sphere at infinity used by Sato [1, 2] and which we shall also consider in Chapters II and III in connection with operators with variable coefficients.

More generally, let $f : \mathbf{R}^n \to \mathbf{R}^N$ be a proper embedding of $\mathbf{R}^n$ in some bounded open set in $\mathbf{R}^N$. Explicitly this means that we assume that $f$ is bounded, continuous and injective, and that the range of $f$ is disjoint from the set of limit points of $f(\xi)$ as $\xi \to \infty$. The closure of $f(\mathbf{R}^n)$ is then a compactification of $\mathbf{R}^n$. We denote it by $W$ and the subset of limit points as $\xi \to \infty$ by $W_0$. Identifying $\mathbf{R}^n$ with $f(\mathbf{R}^n)$ by means of the homeomorphism $f$ we can write $W = W_0 \cup \mathbf{R}^n$ where the union is disjoint and $\mathbf{R}^n$ is a dense open subset.

We make the following important assumptions:

(i) $f$ is semi-algebraic, that is, the graph of $f$ is semi-algebraic;

(ii) $f(\xi+\eta) - f(\xi) \to 0$ as $\xi \to \infty$ if $\eta$ is fixed in $\mathbf{R}^n$.

It is well known that (ii) must be valid uniformly when $|\eta|$ is bounded. In fact, if $\varepsilon > 0$ then

$$E_N = \{ \eta; |f(\xi+\eta) - f(\xi)| < \varepsilon, |\xi| > N \}$$

has positive measure for sufficiently large $N$, and $E_N - E_N$ is then a neighborhood of 0. For $\eta$ in this neighborhood we have $|f(\xi+\eta) - f(\xi)| < 2\varepsilon$ when $|\xi| > N + C$. In view of the assumed pointwise convergence we conclude that (ii) is in fact uniform when $\eta$ is bounded. Using (i) and the Tarski-Seidenberg theorem (see e.g. the appendix in Hörmander [1]) we conclude that for a suitable $K$

$$|f(\xi+\eta) - f(\xi)| < \varepsilon \quad \text{if} \quad |\eta| < \varepsilon^{-1} \quad \text{and} \quad |\xi| \geqq \varepsilon^{-K}.$$

Writing $\delta = 1/K$ we have therefore proved that (i) and (ii) imply

$$|f(\xi+\eta) - f(\xi)| < |\xi|^{-\delta} \quad \text{if} \quad |\eta| < |\xi|^{\delta}. \tag{1.6.1}$$

*Example* 1.6.1. If $f(\xi) = \xi(1+|\xi|^2)^{-1/2}$ the compactification is the unit ball, and $W_0$ is the unit sphere.

All conditions on $f$ are satisfied if we take the direct sum of this $f$ with another $f_1$ satisfying (i) and (ii) only. For $f_1$ we may for example take any quotient $P/\tilde{Q}$ where $Q$ is hypoelliptic and $P$ is weaker than $Q$ (see the proof of Theorem 4.1.6 in Hörmander [1]). Example 1.6.1 is also essentially of this form with $P(\xi) = 1 + |\xi|^2$. Semi-elliptic operators give other useful examples.

For distributions $v \in \mathscr{E}'(\mathbf{R}^n)$ we now introduce the set

$W(v) = W_0 \setminus \{ w \in W_0; \hat{v}(\xi) |\xi|^N$ is bounded for every $N$ in a fixed neighborhood of $w$ in $\mathbf{R}^n \cup W_0 \}$.

Note that if this set is empty, then $\hat{v}(\xi)$ is rapidly decreasing at infinity so $v \in C_0^\infty$.

LEMMA 1.6.2. *If $v \in \mathscr{E}'$ and $\varphi \in C_0^\infty$, then $W(\varphi v) \subset W(v)$.*

*Proof.* Assume that $w \notin W(v)$. This means that for some $\varepsilon > 0$ the Fourier transform $\hat{v}(\xi)$ is rapidly decreasing when $|f(\xi) - w| < \varepsilon$. We claim that the Fourier transform of $v_1 = \varphi v$ is also rapidly decreasing when $|f(\xi) - w| < \varepsilon/2$. Note that when $|f(\xi) - w| < \varepsilon/2$ and $|\xi|$ is large we have $|f(\xi+\eta) - w| < \varepsilon$ if $|\eta| < |\xi|^\delta$, by virtue of (1.6.1). Hence

$$|\hat{v}_1(\xi)| \leqq \int |\hat{v}(\xi-\eta)\hat{\varphi}(\eta)| \, d\eta \leqq C_N |\xi|^{-N} + \\ + C \int_{|\eta| > |\xi|^\delta} |\hat{v}(\xi-\eta)| \, |\hat{\varphi}(\eta)| \, d\eta$$

if $|f(\xi) - w| < \varepsilon/2$. In the last integral we estimate $|\hat{v}(\xi-\eta)|$ by

$C(1+|\xi|)^{\mu}(1+|\eta|)^{\mu}$ where $\mu$ is the order of $v$, and conclude that it is also $O(|\xi|^{-N})$ for every $N$. The proof is complete.

We can now define the wave front set:

*Definition* 1.6.3. If $u \in \mathcal{D}'(X)$ we denote by $WF(u)$ the complement in $X \times W_0$ of the set of all $(x, w)$ such that for some $v \in \mathcal{E}'$ equal to $u$ in a neighborhood of $x$ the Fourier transform of $v$ is rapidly decreasing in a neighborhood of $w$, that is, $w \notin W(v)$.

From the lemma it follows that the fiber of $WF(u)$ over $x$ is the limit of $W(\varphi u)$ when the support of $\varphi$ converges to $x$ while $\varphi(x) \neq 0$. The projection in $X$ of $WF(u)$ is sing supp $u$. In fact, it is trivially included in sing supp $u$. On the other hand, if $x$ is not in the projection of $WF(u)$ it follows by the compactness of $W_0$ and Lemma 1.6.2 that $\varphi u \in C^{\infty}$ for some $\varphi \in C_0^{\infty}$ with $\varphi(x) \neq 0$. Thus we have proved:

THEOREM 1.6.4. *The projection in $X$ of $WF(u)$ is equal to* sing supp $u$.

If $F$ is any closed subset of $X \times W_0$ one can find $u \in C(X)$ with $WF(u) = F$. In fact, since $C_F = \{ u \in C(X), WF(u) \subset F \}$ is a Fréchet space it suffices, in view of the closed graph theorem and Baire's theorem, to show that when $F_1 \subsetneq F_2$ the topologies in $C_{F_1}$ and $C_{F_2}$ are not identical. If $(x_0, w_0) \in F_2 \setminus F_1$ and $\xi_j \in \mathbf{R}^n$ is a sequence with $f(\xi_j) \to w_0$, this follows if we consider a sequence $u(x)\, e^{i<x,\xi_j>}$ where $u \in C_0^{\infty}$ has support close to $x_0$.

The results of section 1.5 can now be improved as follows. For every $w \in W_0$ we introduce the set $L_w(P)$ of all limits of $P_{\xi}/\tilde{P}(\xi)$ as $\xi \to w$. The proof of Theorem 1.5.2 gives the following refinement of Theorem 1.5.1:

THEOREM 1.6.5. *Let $u \in \mathcal{D}'(X)$ where $X$ is an open set in $\mathbf{R}^n$, and let $P(D)u = f \in C^{\infty}(X)$. Assume that $L(P)$ consists of first order operators and let $B_{x,w}$ be the set of all limits of $\Lambda'(Q_j) + \{x\}$ with $Q_j \in L_{w_j}(P)$ and $w_j \to w$. If $(x, w) \in WF(u)$ it follows that for some $b \in B_{x,w}$ the component of $(X \cap b) \times w$ containing $(x, w)$ is also in $WF(u)$.*

The result is particularly satisfactory if $B_{x,w}$ has a unique minimal element. (Note that Theorem 1.6.5 is then equivalent to its local form.) For example, if $P$ is an operator with simple characteristics and $W_0$ is the unit sphere, then $B_{x,w}$ is empty except when $w$ is a characteristic, and $B_{x,w}$ then consists of the corresponding bicharacteristic through $x$. (See example 1.4.4.) It would be interesting to know if for every operator $P$ there is some compactification for which $B_{x,w}$ has a unique minimal element. It may be possible to obtain such results by arguments of the type used by Gabrielov [1] to prove that for every $P$ the closed union of all $\Lambda'(Q)$, $Q \in L(P)$, is a semi-algebraic set of codimension at least one.

For other definitions of the wave front set we refer to Sato [1, 2], and Sato and Kashiwara [1] for the case of hyperfunctions relative to real analytic functions, and to Hörmander [11] for the case of Schwartz distributions relative to any Denjoy-Carleman class of functions which is closed under differentiation and contains the real analytic functions.

# Chapter II

## Some spaces of distributions and operators

### 2.1. *Pseudo-differential operators*

In Chapter I all results ultimately depended on the Fourier transformation. When the coefficients are variable we need to have some substitute. The simplest case occurs in the construction of fundamental solutions for *elliptic* operators with variable coefficients. Classically this was done by perturbation arguments (the E. E. Levi parametrix method, Korn's approximation). These ideas are now embedded in a more manageable and precise form in the theory of pseudo-differential operators.

Let us first note that for an elliptic operator $P(D)$ with constant coefficients of order $m$ we have for some constant $C$,

$$|\xi|^m \leqq C\,|P(\xi)|\,, \quad |\xi| > C\,,$$

if $\xi$ is real or belongs to a narrow cone in $\mathbf{C}^n$ containing $\mathbf{R}^n$. Apart from an integration over a compact set, which contributes an entire analytic term, the fundamental solution constructed in section 1.1 is therefore simply

$$Ef(x) = (2\pi)^{-n} \int e^{i<x,\xi>}\, \chi(\xi)/P(\xi)\hat{f}(\xi)\, d\xi\,.$$

Here $\chi$ is a fixed $C^\infty$ function which is 0 when $|\xi| < C$ and 1 for large $|\xi|$. Differentiation under the sign of integration gives, with $E$ also denoting the distribution such that $Ef = E * f$, (2.1.1) $$P(D)E = \delta + R.$$

Here $\hat{R} = \chi - 1$ so that $R \in C^\infty$. One calls $E$ a *parametrix.* Outside the origin we have $E \in C^\infty$, for if $\alpha$ is large then

$$(-x)^\alpha E = (2\pi)^{-n} \int e^{i<x,\xi>} D_\xi^\alpha \big(\chi(\xi)/P(\xi)\big)\, d\xi\,,$$

and the integrand decreases rapidly at infinity. For the study of regularity properties it is as useful to have a parametrix as to have a fundamental solution: If $v \in \mathscr{E}'$ we obtain $v = E * \big(P(D)v\big) - R * v$. Here $R * v \in C^\infty$ and $E * P(D)v \in C^\infty$ outside sing supp $P(D)v$ since $E \in C^\infty$ outside the origin. This gives

$$\text{sing supp}\, v = \text{sing supp}\, P(D)v$$

when $v$ has compact support and therefore for arbitrary $v$.

Consider now a differential operator $P$ with variable coefficients,

$$P(x, D) = \sum_{|\alpha| \leqq m} a_\alpha(x) D^\alpha$$

in an open set $X \subset \mathbf{R}^n$. We assume that $a_\alpha \in C^\infty(X)$ and that $P$ is elliptic in $X$, that is,

$$P_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x)\, \xi^\alpha \neq 0 \quad \text{if } x \in X \text{ and } 0 \neq \xi \in \mathbf{R}^n.$$

We want to construct a (right) parametrix $E$, that is, a linear map $C_0^\infty(X) \to$ $\to C^\infty(X)$ such that $P(x, D)E = I + R$ where $R$ is an integral operator with $C^\infty$ kernel. The classical method of E. E. Levi is to take a fixed $x_0 \in X$ and try to find $E$ as a perturbation of the known parametrix of the operator $P(x_0, D)$, that is,

$$Ef(x) = (2\pi)^{-n} \int e^{i<x,\xi>}\, \chi(\xi)/P(x_0,\xi) \hat{f}(\xi)\, d\xi\,.$$

Naturally this must be a better approximation at $x_0$ than elsewhere, so the approximation is improved if one replaces $P(x_0, \xi)$ by $P(x, \xi)$. Note that $P(x, \xi)^{-1} = P_m(x, \xi)^{-1} + \ldots$ where dots indicate homogeneous terms of order $-m-1$, $-m-2$, ... . Thus we are led to consider operators of the form

(2.1.2) $$Ef(x) = (2\pi)^{-n} \int e^{i<x,\xi>}\, e(x, \xi) \hat{f}(\xi)\, d\xi$$

where $e$ behaves asymptotically when $\xi \to \infty$ as a sum of homogeneous functions of $\xi$. (See Kohn-Nirenberg [1], Hörmander [2, 4] and the references given there.) Actually it is preferable to make the somewhat less restrictive assumption that $e \in C^\infty(X \times \mathbf{R}^n)$ and that for some $\mu$ and all multi-indices $\alpha$ and $\beta$

$$(2.1.3) \qquad |D_\xi^\alpha D_x^\beta e(x, \xi)| \leqq C_{\alpha,\beta,K}(1+|\xi|)^{\mu-|\alpha|}, \quad x \in K \subset\subset X\,.$$

The set of all such functions will be denoted by $S^\mu(X \times \mathbf{R}^n)$. An operator of the form (2.1.2) with $e \in S^\mu$ is called a *pseudo-differential operator* of order $\mu$ with symbol $e$. It is easy to see that $E$ is a continuous map from $C_0^\infty(X)$ to $C^\infty(X)$ and that $E$ can be extended to a continuous map from $\mathscr{E}'(X)$ to $\mathscr{D}'(X)$. The diagonal in $X \times X$ contains all singularities of the kernel of $E$ (which is a distribution in $X \times X$). Summing up these facts one finds the pseudo-local property

$$(2.1.4) \qquad \text{sing supp}\; Eu \subset \text{sing supp}\; u, u \in \mathscr{E}'(X)\,.$$

To complete the construction of a parametrix for the elliptic operator $P$ it suffices to choose $e$ so that

$$P(x, D+\xi)\,e(x,\xi) - 1 \in S^{-\infty} = \bigcap_\mu S^\mu\,.$$

To do so we choose $e$ asymptotic to a sum $e_0 + e_1 + \ldots$ where $e_j$ is homogeneous of degree $-m-j$ with respect to $\xi$ and

$$P(x, D+\xi)(e_0 + \ldots + e_j) - 1 \in S^{-j-1}, \; j = 0, 1, \ldots\,.$$

This means for $j = 0$ that $e_0 = 1/P_m$. Since $P(x, D+\xi)\,e_j - P_m(x,\xi)\,e_j \in$ $\in S^{-j-1}$ the conditions are recursively satisfied by a suitable choice of $e_j$. This formal successive approximation is of course just a simpler way of carrying out the classical iterative procedures for solving the integral equations which occur in the E. E. Levi method. It is more appropriate though, since it avoids strict convergence requirements which force one to work locally only.

Pseudo-differential operators not only give a convenient framework for the construction of parametrices for elliptic equations but they form a natural extension of the class of differential operators. A differential operator $P(x, D)$ is obviously of the form (2.1.2) with $e(x, \xi) = P(x, \xi)$. It turns out that also pseudo-differential operators form an *algebra which is invariant under passage to adjoints and changes of variables*; the latter fact immediately allows an extension of the definition to manifolds. The usual formulas of calculus remain valid with obvious modifications. For example, if $P =$ $= P(x, D)$ and $Q = Q(x, D)$ are differential operators then the symbol of the differential operator $R = QP$ is given by

$$(2.1.5) \qquad R(x, \xi) = \sum ((iD_\xi)^\alpha Q(x, \xi)) D_x^\alpha P(x, \xi)/\alpha\,!$$

If $P$ and $Q$ are pseudo-differential operators with symbols $P(x, \xi)$, $Q(x, \xi)$ the product $R = QP$ is again a pseudo-differential operator and for the symbol $R(x, \xi)$ the formula (2.1.5) is valid *mod* $S^\mu$ for every $\mu$, which makes sense since all but a finite number of terms are in $S^\mu$. One precaution must be made though, for to compose pseudo-differential operators we must assume that they map $C_0^\infty$ to $C_0^\infty$, and preferably also $C^\infty$ to $C^\infty$. Since the kernel of a pseudo-differential operator is in $C^\infty$ outside the diagonal in $X \times X$ it can be modified without changing the singularities to a kernel $K$ with support so close to the diagonal that the projections $\operatorname{supp} K \to X$ are both proper. This implies the desired properties. We shall say that an operator with such a kernel is *properly supported.* By $L^\mu(X)$ we denote the space of properly supported pseudo-differential operators of order $\mu$. The definition is clearly valid also if $X$ is a $C^\infty$ manifold.

Generalizing a definition in section 1.3 for differential operators we shall say that a pseudo-differential operator $P$ of order $m$ with symbol $p$ is characteristic at $(x, \xi) \in X \times (\mathbf{R}^n \backslash 0)$ if

$$\varliminf_{t \to +\infty} \quad | p(x, t\,\xi) |\, t^{-m} = 0\,.$$

The characteristic points form a closed cone in $X \times (\mathbf{R}^n \backslash 0)$ which regarded as a subset of $T^*(X)\backslash 0$ is invariant under a change of variables and therefore well defined even if $X$ is a manifold. If no characteristic exists, we say that $P$ is *elliptic.* The arguments above show that if $P$ is elliptic of order $m$ one can find $Q$ elliptic of order $-m$ so that $QP - I = R_1$ and $PQ - I = R_2$ have $C^\infty$ kernels. This shows that also for elliptic pseudo-differential operators we have

$$\operatorname{sing\,supp} u = \operatorname{sing\,supp} Pu, \; u \in \mathscr{D}'(X)\,.$$

The construction of fundamental solutions in section 1.1 also simplifies very much when $P$ is just hypoelliptic, that is, $P$ satisfies (1.4.2). This condition implies that $P(\xi) \neq 0$ for large $\xi$ and that for some $\rho > 0$

$$| D_\xi^\alpha P(\xi) | / | P(\xi) | \leqq C\, | \xi |^{-\rho|\alpha|}\,,$$

One can still find a parametrix of the form (2.1.2), but $e(x, \xi) = 1/P(\xi)$ satisfies a weaker condition than (2.1.3). One is therefore led to introduce the set $S^m_{\rho,\delta}$ of functions such that for all multi-indices

$$(2.1.3)' \quad | D_\xi^\alpha D_x^\beta e(x, \xi) | \leqq C_{\alpha,\beta,K}(1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|}, \; x \in K \subset\subset X\,.$$

When $0 \leqq \delta < \rho \leqq 1$ one obtains again a self adjoint algebra of operators, and it is invariant under a change of variables if in addition $1 - \rho \leqq \delta$. If for some $\delta < \rho$

$$|D_\xi^\alpha D_x^\beta p(x, \xi)| / |p(x, \xi)| \leqq C_{\alpha,\beta,K}(1 + |\xi|)^{-\rho|\alpha|+\delta|\beta|}, \; x \in K ((X,$$

and $1/|p(x, \xi)| \leqq C|\xi|^M$ for some $M$, one can as in the elliptic case construct a parametrix of the same type and conclude that the operator with symbol $p$ is hypoelliptic. (See Hörmander [4] and for the case of systems also Hörmander [8].) However, for the sake of brevity we shall ignore extensions of this type in what follows.

If $L^2_{loc}(X)$ is the set of functions in $X$ which are square integrable on compact subsets of every coordinate patch (with the obvious topology), then every $P \in L^0(X)$ is a continuous map $L^2_{loc}(X) \to L^2_{loc}(X)$. If we define $H_{(s)}(X)$ to be the set of all distributions such that $Pu \in L^2_{loc}(X)$ when $P \in L^s(X)$ it follows that $H_{(0)}(X) = L^2_{loc}(X)$, and that $L^m(X)$ maps $H_{(s)}(X)$ continuously into $H_{(s-m)}(X)$. Conversely $Pu \in H_{(s-m)}(X)$ implies $u \in H_{(s)}(X)$ if $P \in L^m(X)$ is elliptic, so $u \in H_{(s)}(X)$ if (and only if) $Pu \in L^2_{loc}(X)$ for some elliptic $P$ of order $s$. Similar definitions can be made with $L^2$ replaced by $L^p$ if $1 < p < \infty$.

## 2.2. *The wave front set*

If $u \in \mathscr{D}'(X)$ we have by definition

$$\text{sing supp } u = \cap \{x; \varphi(x) = 0\}$$

the intersection being taken over all $\varphi \in C^\infty(X)$ with $\varphi u \in C^\infty(X)$. Replacing the function $\varphi$ by a pseudo-differential operator $A$ we introduce

$$WF(u) = \bigcap_{Au \in C^\infty} char(A) \tag{2.2.1}$$

where *char* $(A)$ is the set of characteristics of $A$. It is clear that this is a closed cone in $T^*(X)\backslash 0$ with projection in $X$ contained in sing supp $u$. In fact, it is equal to sing supp $u$ for if $x$ is not in the projection of $WF(u)$ we can find finitely many operators $A_j \in L^0$ with $A_j u \in C^\infty$ so that $T_x^* \cap (\cap \, char(A_j)) = \emptyset$. If $A = \Sigma A_j^* A_j$ we have $Au \in C^\infty$ and $A$ is elliptic at $x$ so $u \in C^\infty$ there. Thus we have

THEOREM 2.2.1. *The projection of $WF(u)$ in $X$ is equal to* sing supp $u$.

We shall call $WF(u)$ the *wave front set* of $u$. (The relation to the definitions in section 1.6 will be discussed after Theorem 2.2.3.) Clearly it describes the location of the singularities and the frequencies which occur in their harmonic decomposition. The definition we have given leads immediately to a regularity theorem for any pseudo-differential operator:

THEOREM 2.2.2. *If $A$ is a pseudo-differential operator then*

$$WF(Au) \subset WF(u) \subset WF(Au) \cup char(A). \tag{2.2.2}$$

*Proof.* The second part, extending the regularity theorem for elliptic operators is obvious, but the first which improves the pseudolocal property (2.1.4) may require some comment. We may assume that $X \subset \mathbf{R}^n$ since the definition of $WF(u)$ is local in $X$. For any $(x_0, \xi_0) \notin WF(u)$ we can choose a pseudo-differential operator $B$ which is non-characteristic at $(x_0, \xi_0)$ so that $Bu \in C^\infty$. If $C$ is a pseudo-differential operator whose symbol is of order $-\infty$ outside a small conic neighborhood of $(x_0, \xi_0)$ we can find another operator $C_1$ such that $CA = C_1 B$, by multiplying $CA$ to the right with the formal inverse of $B$ which exists near $(x_0, \xi_0)$. Thus $CAu = C_1 Bu \in$ $\in C^\infty$ and we conclude that $(x_0, \xi_0) \notin WF(Au)$.

In the definition of the wave front set it is easily seen that one can restrict oneself to operators $A$ of order 0 and even operators of the form $b(D)\, a(x)$ where $b(\xi)$ is a homogeneous function of degree 0 for large $|\xi|$. This leads to an equivalent definition which is more useful in many proofs:

THEOREM 2.2.3. $(x_0, \xi_0) \notin WF(u)$ *if and only if for some coordinate patch containing $x_0$ one can find $v \in \mathscr{E}'$ equal to $u$ in a neighborhood of $x_0$ and with $\hat{v}(\xi) = 0(|\xi|^{-N})$ for every $N$ in a conic neighborhood of $\xi_0$ independent of $N$.*

The theorem shows that $WF(u)$ regarded as a subset of the sphere bundle agrees with the set given by Definition 1.6.3 when $X \subset \mathbf{R}^n$ and $W_0$ is the compactification by a sphere. The definition used here has the advantage that the invariance under a change of variables follows from the invariance of pseudo-differential operators.

We shall now list a number of properties of wave front sets. Most of them are due to Sato who considered hyperfunctions modulo real analytic functions. (See Sato [2], Sato-Kawai [1], and Sato-Kashiwara [1].) For complete proofs using Theorem 2.2.3 see Hörmander [9, section 2.5].

First we consider the product of two distributions $u_1$ and $u_2$. Let $\chi \in C_0^\infty(\mathbf{R}^n)$, $\int \chi dx = 1$, and set $\chi_\varepsilon(x) = \varepsilon^{-n}\chi(x/\varepsilon)$. Assuming that $u_j \in \mathscr{E}'(\mathbf{R}^n)$ we wish to define $u_1 u_2$ as the limit of $(u_1 * \chi_\varepsilon)(u_2 * \chi_\varepsilon)$ as $\varepsilon \to 0$. In general this is not possible but the limit does exist if

$$(2.2.3) \qquad WF(u_1) + WF(u_2) = \{(x, \xi_1 + \xi_2);\ (x, \xi_j) \in WF(u_j)\} \subset T^*(X)\backslash 0$$

It is then independent of the choice of coordinates and $\chi$. The situation is summed up in

THEOREM 2.2.4. *If $u_1, u_2 \in \mathscr{D}'(X)$ and (2.2.3) is fulfilled, there is a natural way of defining $u_1 u_2$ and we have*

$$(2.2.4) \qquad WF(u_1 u_2) \subset WF(u_1) \cup WF(u_2) \cup \big(WF(u_1) + WF(u_2)\big).$$

*Here the right hand side is closed and X may be a manifold.*

With suitable definitions the multiplication is continuous when introduced in this way. In the following theorems the word " natural " will refer to a definition by continuity as in Theorem 2.2.4.

THEOREM 2.2.5. *Let X and Y be manifolds and $\varphi : Y \to X$ a $C^\infty$ map. Let $u \in \mathscr{D}'(X)$ and assume that*

$$\varphi^* WF(u) = \{(y, {}^t\varphi_y'(y)\,\xi), (\varphi(y), \xi) \in WF(u)\} \subset T^*(Y)\backslash 0.$$

*Then there is a natural way of defining the composition $\varphi^* u$ of u with $\varphi$ so that it is the standard composition when u is a function. We have*

$$(2.2.5) \qquad WF(\varphi^* u) \subset \varphi^* WF(u).$$

Note that the pullback $\varphi^* u$ is defined for all $u \in \mathscr{D}'(X)$ precisely when $\varphi'$ is surjective, and then it is well known that such a definition is possible. In particular we see that if $Y \subset X$ is a submanifold, we can define the restriction of $u$ to $Y$ if the normal bundle $N(Y)$ does not meet $WF(u)$. For example, if $u \in \mathscr{D}'(X)$ and $Au \in C^\infty$ for some pseudo-differential operator $A$, we can define the restriction of $u$ to $Y$ if $Y$ is non-characteristic, that is, the normals to $Y$ are non-characteristic with respect to $A$. This is also a well known fact (partial hypoellipticity).

Let $X$ and $Y$ be two $C^\infty$ manifolds with given positive $C^\infty$ densities. By the kernel theorem of Schwartz we can then identify $\mathscr{D}'(X \times Y)$ with the space of continuous linear operators $C_0^\infty(Y) \to \mathscr{D}'(X)$ by means of the formula

$$< K\varphi, \psi > \ = K(\psi \otimes \varphi); \ \ \varphi \in C_0^\infty(Y), \psi \in C_0^\infty(X);$$

on the right $K$ denotes an element of $\mathscr{D}'(X \times Y)$ and on the left the corresponding linear transformation. In terms of the wave front set of $K$ we can state useful sufficient conditions for regularity of $K$ in the sense of Schwartz [1]:

THEOREM 2.2.6. *For any $u \in C_0^\infty(Y)$ the set*

$$(2.2.6) \quad WF_X(K) = \{(x, \xi); (x, \xi, y, 0) \in WF(K) \text{ for some } y \in Y\}$$

*contains $WF(Ku)$. Thus $K$ maps $C_0^\infty(Y)$ into $C^\infty(X)$ if $WF_X(K) = \varnothing$, that is, if $WF(K)$ contains no point which is normal to a manifold $x$ = constant.*

THEOREM 2.2.7. *$Ku$ can be defined in a natural way when $u \in \mathscr{E}'(Y)$ and $WF(u)$ does not meet the set*

$$(2.2.7) \quad WF'_Y(K) = \{(y, \eta); (x, 0, y, -\eta) \in WF(K) \text{ for some } x \in X\}.$$

*Thus $K$ can be extended to a continuous map $\mathscr{E}'(Y) \to \mathscr{D}'(X)$ if $WF'_Y(K) = \varnothing$, that is, $WF(K)$ contains no point which is normal to a manifold $y$ = constant.*

The proof of Theorem 2.2.6 follows easily from the description of the wave front set given in Theorem 2.2.3. Theorem 2.2.7 follows by duality.

If we have three manifolds $X$, $Y$, $Z$ and distributions $K_1 \in \mathscr{D}'(X \times Y)$, $K_2 \in \mathscr{D}'(Y \times Z)$ where for simplicity we assume that $K_1$ and $K_2$ are properly supported, then $K_2 u \in \mathscr{E}'(Y)$ and $WF(K_2 u) \subset WF_Y(K_2)$ when $u \in C_0^\infty(Z)$. The composition $K_1(K_2 u)$ is therefore defined if

$$(2.2.8) \qquad WF'_Y(K_1) \cap WF_Y(K_2) = \varnothing,$$

and it is of the form $(K_1 \circ K_2)u$ where $K_1 \circ K_2 \in \mathscr{D}'(X \times Z)$. When writing down an inclusion for the wave front set of $K_1 \circ K_2$ it is convenient to introduce for example

$$WF'(K_1) = \{(x, \xi, y, \eta); (x, \xi, y, -\eta) \in WF(K_1)\},$$

that is, multiply by $-1$ in the fiber of the second tangent space involved.

THEOREM 2.2.8. *When* (2.2.8) *is fulfilled we have*

$$(2.2.9)\qquad WF'(K_1 \circ K_2) \subset \big(WF'(K_1) \circ WF'(K_2)\big) \cup \big(WF_X(K_1) \times Z\big) \cup \big(X \times WF'_Z(K_2)\big).$$

*Here* $WF'(K_1)$ *and* $WF'(K_2)$ *are composed as relations from* $T^*(Y)$ *to* $T^*(X)$ *and from* $T^*(Z)$ *to* $T^*(Y)$. *The right hand side of* (2.2.9) *is closed.*

The special case when $Z$ reduces to a point is worth special notice:

THEOREM 2.2.9. *Let* $K \in \mathscr{D}'(X \times Y)$ *and* $u \in \mathscr{E}'(Y)$, $WF(u) \cap WF'_Y(K) = \varnothing$. *Then we have*

$$(2.2.10)\qquad WF(Ku) \subset \big(WF'(K) \circ WF(u)\big) \cup WF_X(K)$$

*where again* $WF'(K)$ *is interpreted as a relation mapping sets in* $T^*(Y)$ *to sets in* $T^*(X)$.

In section 2.3 we shall describe the wave front set for some important classes of distributions. In preparation for this we shall now discuss how the wave front set can be used to localize various spaces of distributions not only in $X$ but in $T^*(X)\backslash 0$ (or rather the cosphere bundle $S^*(X)$ which is the quotient by the multiplicative group of positive reals).

Let $\mathscr{F}$ be a linear subspace of $\mathscr{D}'(X)$. If $x_0 \in X$ we shall say that a distribution $u$ in $X$ belongs to $\mathscr{F}$ at $x_0$ if one can find $v \in \mathscr{F}$ so that $v - u = 0$ in a neighborhood of $x_0$. We call $\mathscr{F}$ *local* if every distribution which belongs to $\mathscr{F}$ at every $x_0 \in X$ is in fact in $\mathscr{F}$. (This means that $\mathscr{F}$ is the space of sections of the sheaf of germs of sections of $\mathscr{F}$.)

If $(x_0, \xi_0) \in T^*(X)\backslash 0$ we shall say that $u \in \mathscr{F}$ at $(x_0, \xi_0)$ if one can find $v \in \mathscr{F}$ so that $(x_0, \xi_0) \notin WF(u-v)$. Repeating the proof of Theorem 2.2.1 one shows that when $C^\infty(X) \subset \mathscr{F}$ and $\mathscr{F}$ is an $L^0$ *module*, then $u \in \mathscr{F}$ at $x_0$ if (and only if) $u \in \mathscr{F}$ at $(x_0, \xi_0)$ for every $\xi_0 \in T^*_{x_0}\backslash 0$. If in addition $\mathscr{F}$ is local we therefore conclude that $u \in \mathscr{F}$ if and only if $u \in \mathscr{F}$ at $(x_0, \xi_0)$ for all $(x_0, \xi_0) \in T^*(X)\backslash 0$. As an example of this we may take $\mathscr{F} = H_{(s)}(X)$.

We can also piece together spaces of distributions from local data. Let $\{U_i\}_{i \in I}$ be a covering of $T^*(X)\backslash 0$ by open cones and let $\mathscr{F}_i$, $i \in I$, be an $L^0$ submodule of $\mathscr{D}'(X)$ containing $C^\infty(X)$. Assume that if $(x_0, \xi_0) \in U_i \cap U_j$ then every element of $\mathscr{F}_i$ is in $\mathscr{F}_j$ at $(x_0, \xi_0)$. If we set

$$\mathscr{F} = \{u \in \mathscr{D}'(X);\, u \in \mathscr{F}_j \text{ at every point in } U_j \text{ for all } j\}$$

we obtain a local $L^0$ module of distributions. If $(x_0, \xi_0) \in U_j$ we have $u \in \mathscr{F}$ at $(x_0, \xi_0)$ if and only if $u \in \mathscr{F}_j$ at $(x_0, \xi_0)$.

## 2.3. *Distributions defined by Fourier integrals*

If in (2.1.2) we introduce the definition of the Fourier transformation we see formally that the distribution kernel of the pseudo-differential operator $E$ is given by

$$(x, y) \to (2\pi)^{-n} \int e^{i<x-y,\theta>} e(x, \theta)\, d\theta\,. \tag{2.3.1}$$

Similarly the fundamental solution of the wave equation $\partial^2 u/\partial t^2 - \Delta u = 0$ in $n$ space variables $(n > 1)$ with pole at $(y, 0)$ is at time $t > 0$ given by

$$(x, y) \to (2\pi)^{-n} \left(\int e^{i(<x-y,\theta>+t|\theta|)} (2i|\theta|)^{-1}\, d\theta - \int e^{i(<x-y,\theta>-t|\theta|)} (2i|\theta|)^{-1}\, d\theta\right). \tag{2.3.2}$$

These examples suggest the importance of the classes of distributions which we shall study now.

Let $X \subset \mathbf{R}^n$ and let $\Gamma$ be an open cone in $X \times (\mathbf{R}^N \backslash 0)$ for some $N$. Assume given a function $\varphi \in C^\infty(\Gamma)$ satisfying the following conditions:

(i) $\varphi$ is positively homogeneous with respect to the variables in $\mathbf{R}^N$.

(ii) $\operatorname{Im} \varphi \geqq 0$.

(iii) $d\varphi \neq 0$ everywhere in $\Gamma$.

Such a function will be called a *phase* function. Let $S_0^m(\Gamma)$ be the set of all $a \in S^m(X \times (\mathbf{R}^N \backslash 0))$ (see section 2.1) vanishing in a conic neighborhood of $\complement\Gamma$.

For $a \in S_0^m(\Gamma)$ we claim that the integral

$$A(x) = \int e^{i\varphi(x,\theta)} a(x, \theta)\, d\theta \tag{2.3.3}$$

can be defined, not necessarily as a function of $x$ but as a distribution in $X$. To do so we consider the linear form

$$I(u) = \iint e^{i\varphi(x,\theta)} a(x, \theta)\, u(x)\, dx d\theta, u \in C_0^\infty(X)\,. \tag{2.3.4}$$

In view of (iii) the fact that

$$e^{i\varphi} = D(e^{i\varphi})/(Di\,\varphi)$$

allows one, by successive (formal) partial integrations with no boundary terms, to reduce the growth of the integrand at infinity until it becomes integrable. This gives a precise definition of $I(u)$ and the linear form $u \to I(u)$ is then a distribution $A \in \mathscr{D}'(X)$. If $\chi \in \mathscr{S}(\mathbf{R}^N)$, $\chi(0) = 1$, it is easily shown that

$$(2.3.5) \qquad A = \lim_{\varepsilon \to 0} \int e^{i\varphi(.,\theta)} \chi(\varepsilon\theta)\, a(.,\theta)\, d\theta$$

with the limit in the weak topology of $\mathscr{D}'(X)$. Thus the definition of (2.3.4) by partial integrations is quite independent of how these are carried out, and (2.3.5) is independent of the choice of $\chi$. We shall call (2.3.3) an oscillatory integral but use the standard notation. (For these facts as well as most of this section we refer to Hörmander [9].) The integral (2.3.3) is thus defined for a fixed $x = x_0$ if $\varphi(x_0, \theta)$ has no critical point $(x_0, \theta) \in \Gamma$ as a function of $\theta$. In that case, $A \in C^\infty$ near $x_0$. Note that if $(x_0, \theta)$ is a critical point of $\varphi$ as a function of $\theta$, then $\varphi(x_0, \theta) = 0$ by Euler's identity for homogeneous functions. On the other hand, when $\varphi(x_0, \theta) = 0$ it follows from (ii) that $d(\operatorname{Im} \varphi(x, \theta)) = 0$ so $d_{x,\theta} \operatorname{Re} \varphi(x, \theta) \neq 0$ by (iii).

To determine the wave front set of $A$ we use Theorem 2.2.3. Thus we take a function $u \in C_0^\infty$ equal to 1 near $x_0$ and with small support, and study

$$< A, ue^{-i<x,\xi>} > = \iint e^{i(\varphi(x,\theta) - <x,\xi>)}\, u(x)\, a(x, \theta)\, dx d\theta$$

as $\xi \to \infty$ in a conic neighborhood of $\xi_0$ (oscillatory integral !). Naturally the main contributions come from critical points in the exponent, that is, points where $\varphi_\theta' = 0$, $\varphi_x' = \xi$. Indeed, we have

THEOREM 2.3.1. *If $A$ is defined by* (2.3.5) *then*

$$(2.3.6) \quad WF(A) \subset \{ (x, \varphi_x'(x, \theta)); (x, \theta) \in \Gamma \text{ and } \varphi_\theta'(x, \theta) = 0 \} \subset T^*(X) \backslash 0$$

*In particular,*

$$(2.3.7) \quad \text{sing supp } A \subset \{ x; \varphi_\theta'(x, \theta) = 0 \text{ for some } \theta \text{ with } (x, \theta) \in \Gamma \}.$$

As an example we see from (2.3.1) that the wave front set of the kernel of a pseudo-differential operator $E$ lies in $\{ (x, y; \xi, \eta); x = y, \xi = -\eta \}$ which is the normal bundle of the diagonal. Thus $WF'(E)$ is in the diagonal of $T^*(X) \times T^*(X)$, which allows us to identify the wave front set of a pseudo-differential operator in $X$ with a closed cone in $T^*(X) \backslash 0$. In view of Theorem 2.2.9 this result contains the left hand part of (2.2.2) (the improved pseudo-local property).

As a second example we see that for the two terms in (2.3.2) the wave front set lies in the set where $x - y = \mp t\theta / |\theta|$ and $\xi = -\eta = \theta$.

This corresponds to the two components of the normal bundle of $\{(x, y); |x - y|^2 = t^2\}$. In particular the singularities are carried by the light cone.

The set of distributions which can be written in the form (2.3.5) with a given $\varphi$ and $a \in S_0^m(\Gamma)$ is always an $L^0(X)$ module. (For a proof when $\varphi$ is real see Theorem 2.12 in Hörmander [6].) We can therefore use the remarks at the end of section 2.2 to define global spaces of distributions which locally in $T^*(X)\backslash 0$ have such representations.

We shall restrict ourselves in what follows to the case where $\varphi$ is *real and non-degenerate*, that is, the differentials of the functions $\partial\varphi/\partial\theta_j$ are linearly independent in $C = \{(x, \theta) \in \Gamma; \varphi'_\theta(x, \theta) = 0\}$. The map

$$C \ni (x, \theta) \to (x, \varphi'_x) \tag{2.3.8}$$

to the wave front set has an injective differential then. The range $\Lambda$ is locally a conic manifold in $T^*(X)\backslash 0$ of dimension *dim* $X$. Let $(x, \xi)$ denote the standard coordinates in $T^*(X)$ obtained from local coordinates $x_1, \ldots, x_n$ in $X$ by taking $dx_1, \ldots, dx_n$ as basis for the cotangent vectors. The form $\Sigma\xi_j\, dx_j$ is then invariantly defined, and the restriction to $\Lambda$ is $\varphi'_x\, dx = d\varphi - \varphi'_\theta\, d\theta = 0$. In view of the homogeneity this is equivalent to the vanishing on $\Lambda$ of the differential which is the symplectic form $\sigma = \Sigma d\xi_j \wedge dx_j$. Submanifolds of $T^*(X)$ of dimension *dim* $X$ on which the symplectic form vanishes also play a fundamental role in the classical integration theory of first order differential equations (see section 3.1). Following Maslov [1] we shall call them *Lagrangean manifolds.*

Locally the class of distributions which can be written in the form (2.3.3) for some $a \in S_0^{m+n/4-N/2}(\Gamma)$, $n = \dim X$, and a non-degenerate real phase function $\varphi$ depends only on the Lagrangean manifold $\Lambda$ corresponding to $\varphi$ and on no other properties of this function. Any closed conic Lagrangean submanifold $\Lambda \subset T^*(X)\backslash 0$ (or a closed conic subset of a Lagrangean submanifold which is not necessarily closed) can locally be represented as the range of a map (2.3.8). We can therefore define a space $I^m(X, \Lambda)$ of distributions with wave front set in $\Lambda$ which locally can be written in the form (2.3.3) with $a \in S_0^{m+n/4-N/2}$ and $\varphi$ defining a part of $\Lambda$ according to (2.3.8). With the elements in $I^m(X, \Lambda)$ one can, as for pseudo-differential operators, associate principal symbols on $\Lambda$, which are symbols of order $m + n/4$ modulo symbols of order $m + n/4 - 1$ (with values in certain line bundles). The notion of characteristic point can therefore be defined as in section 2.1. For the kernels of pseudo-differential operators in $X$ which are associated with the normal bundle of the diagonal in $X \times X$

this agrees with the standard notion of principal symbol. (Note that the normal bundle of any submanifold $Y$ of $X$ is Lagrangean in $T^*(X)$ and that the normal bundle of the diagonal in $X \times X$ can be identified with $T^*(X)$.)

When we take a conic Lagrangean submanifold of $T^*(X \times Y)\backslash 0$ where $X$ and $Y$ are two manifolds we can interpret the distributions in $I^m(X \times Y, \Lambda)$ as maps from $C_0^\infty(Y)$ to $\mathscr{D}'(X)$. When $\Lambda \subset \big(T^*(X)\backslash 0\big) \times \times \big(T^*(Y)\backslash 0\big)$ we have seen (Theorems 2.2.6 and 2.2.7) that they are actually continuous operators from $C_0^\infty(Y)$ to $C^\infty(X)$ and from $\mathscr{E}'(X)$ to $\mathscr{D}'(Y)$. The set

$$\Lambda' = \{(x, \xi, y, -\eta);\ (x, \xi, y, \eta) \in \Lambda\}$$

will then be called a homogeneous *canonical relation*; it is Lagrangean with respect to the symplectic form $\sigma_X - \sigma_Y$. This is the set which occurs in the multiplicative properties of wave front sets described in Theorem 2.2.8. If we have three manifolds $X$, $Y$, $Z$ and canonical relations $C_1$, $C_2$ from $T^*(Y)$ to $T^*(X)$ resp. $T^*(Z)$ to $T^*(Y)$ one can supplement Theorem 2.2.8 by proving that the composition $K_1 \circ K_2$ of properly supported operators

$$K_1 \in I^{m_1}(X \times Y, C_1') \text{ and } K_2 \in I^{m_2}(Y \times Z, C_2')$$

is in

$$I^{m_1+m_2}(X \times Z, (C_1 \circ C_2)')$$

if the appropriate transversality and other conditions are fulfilled which guarantee that $C_1 \circ C_2$ is a manifold. There is a simple formula giving the principal symbol of $K_1 \circ K_2$ as a product of those of $K_j$. (The normalization of the degree for operators in $I^m$ was chosen precisely to make the preceding statement valid.) For complete statements and proofs we refer to Hörmander [9]; a summary is given in Hörmander [10]. However, we shall consider an important special case due to Egorov [1] which gave rise to much of the work described here.

Thus assume that $X$ and $Y$ have the same dimension and that $\Lambda'$ is the graph of a homogeneous *canonical transformation* $\chi$ from $T^*(Y)$ to $T^*(X)$ (or only a local canonical transformation in which case we consider a closed conic subset). That $\chi$ is canonical means that $\chi^*\sigma_X - \sigma_Y = 0$ or that $\sigma_X - \sigma_Y$ vanishes on $\Lambda'$, so we have a canonical relation in the sense explained above. If $K \in I^m(X \times Y, \Lambda)$, then the adjoint $K^*$ belongs to the inverse transformation and the compositions $KK^*$ and $K^*K$ belong to the identity, that is, they are pseudo-differential operators in $X$ and in $Y$

respectively. If $A$ is a pseudo-differential operator in $X$ of order $\mu$ then the product $AK$ is in $I^{m+\mu}(X \times Y, \Lambda)$ and the principal symbol is the product of the principal symbol of $K$ (considered as living on $\Lambda'$) by that of $A$ lifted from $T^*(X)$ to $\Lambda'$ by the projection $\Lambda' \to T^*(X)$. If we multiply to the right instead the result is the same except that we shall use the projection from $\Lambda'$ to $T^*(Y)$. If $A$ and $B$ are pseudo-differential operators in $X$ and in $Y$ respectively and if $AK = KB$ we conclude that for the principal symbols $a$ and $b$ of $A$ and $B$ we must have

$$a(\chi(y, \eta)) = b(y, \eta) \tag{2.3.8}$$

if the principal symbol of $K$ is not 0 at $(\chi(y, \eta), (y, -\eta))$. Conversely, (2.3.8) implies that $AK - KB$ is of lower order. We can therefore successively construct the symbol of $B$ for a given $A$ so that $AK - KB$ is of order $-\infty$, provided that the wave front set of $A$ is concentrated near a point where $K$ is elliptic. This argument often allows one to pass from one operator to another with principal symbol modified by a homogeneous canonical transformation. (See also Lemma 3.2.2 below.)

The operators in $I^m(X \times Y, \Lambda')$ can be described by means of the classical *generating function*: For any point $(x_0, \xi_0, y_0, \eta_0)$ in the graph of $\chi$ one can choose local coordinates in neighborhoods of $x_0$ and $y_0$ so that there is a function $\varphi(x, \eta)$ in a conical neighborhood of $(x_0, \eta_0)$ which is homogeneous of degree 1 with respect to $\eta$, such that $\chi$ is given by $(\varphi'_\eta, \eta) \to \to (x, \varphi'_x)$ and $det\, \varphi''_{x\eta} \neq 0$. The elements $A$ in $I^m(X \times Y, \Lambda)$ with wave front set close to $(x_0, \xi_0, y_0, -\eta_0)$ are then as operators of the form

$$Au(x) = (2\pi)^{-n} \int e^{i\varphi(x,\eta)} a(x, \eta)\, \hat{u}(\eta)\, d\eta, \quad a \in S^m(X \times \mathbf{R}^n),$$

when $u$ is in $C_0^\infty$ in a neighborhood of $y_0$ and $x$ is in a neighborhood of $x_0$. The assertions made above are easy to prove directly from this representation.

# Chapter III

## PSEUDO-DIFFERENTIAL OPERATORS WITH NON-SINGULAR CHARACTERISTICS

### 3.1. *Preliminaries*

Throughout this chapter $X$ will denote a $C^\infty$ manifold (all manifolds are tacitly assumed countable at infinity) and $P$ a properly supported pseudo-differential operator in $X$ of order $\mu$ with homogeneous principal symbol $p$. This means that $p$ is a complex valued $C^\infty$ homogeneous function of degree $\mu$ on $T^*(X)\backslash 0$ and that for every local coordinate system the full symbol of $P$ differs from $p$ by a symbol in $S^{\mu-1}$. We shall also require that the characteristics are simple, that is,

$$dp(x, \xi) \neq 0 \text{ if } (x, \xi) \in T^*(X)\backslash 0 \text{ and } p(x, \xi) = 0 . \tag{3.1.1}$$

The purpose is to give analogues of the existence theorems stated in Chapter I for the case of differential operators with constant coefficients, in particular part iii) of Theorem 1.4.6 and the related Theorems 1.5.1 and 1.5.2. This will require further conditions on $P$ which will be introduced later on.

We shall now recall some classical facts concerning the integration of the first order differential equation

$$p(x, \operatorname{grad} u) = 0 . \tag{3.1.2}$$

At first it will be assumed that $p$ is real valued. If $u \in C^2(Y)$ for an open set $Y \subset X$ and if $u$ is real valued, then $\Lambda = \{(x, \operatorname{grad} u(x)), x \in Y\}$ is a *section* of $T^*(X)$ over $Y$ on which (the restriction of) the invariant symplectic form $\sigma_X = \Sigma d\xi_j \wedge dx_j$ vanishes. In fact, the pullback of $\sigma_X$ to $Y$ by the section is

$$d(\Sigma \partial u/\partial x_j dx_j) = d\,du = 0 .$$

Conversely, if we have a $C^1$ section $\Lambda$ of $T^*(X)$ over $Y$ on which $\sigma_X$ vanishes, we can define $\Lambda$ in local coordinates by $\xi = \xi(x)$, and $\partial\xi_j/\partial x_k - \partial\xi_k/\partial x_j = 0$ so $\xi = du$ for some function $u$ in $Y$ (determined up to an additive constant) if $Y$ is simply connected. The (local) integration of (3.1.2) is therefore equivalent to finding a (local) section $\Lambda$ of $T^*(X)$ such that

(i) $\sigma = 0$ on $\Lambda$

(ii) $p = 0$ on $\Lambda$.

In other words, $\Lambda$ shall be a Lagrangean manifold (see section 2.3) contained in $p^{-1}(0)$ such that the projection $\Lambda \to X$ is a diffeomorphism. Locally the last condition means just that $\Lambda$ is transversal to the fiber of the projection $T^*(X) \to X$. In the local theory one can therefore concentrate on (i) and (ii).

The symplectic form $\sigma$ is a non-degenerate skew symmetric bilinear form on the tangent space of $T^*(X)$. That a manifold $\Lambda$ is Lagrangean therefore means that at every point $\lambda \in \Lambda$ the tangent space $T_\lambda(\Lambda)$ is its own orthogonal complement with respect to $\sigma$. If (ii) is valid we have $dp = 0$

on $T_\lambda(\Lambda)$. The tangent vector $H_p$ to $T^*(X)$ corresponding to the covector $dp$ by the definition

$$< t, dp > \; = \sigma(t, H_p), t \in T(T^*(X)),$$

is therefore tangential to $\Lambda$. One calls $H_p$ the Hamiltonian vector field defined by $p$. In terms of local coordinates $x$ in $X$ and the corresponding coordinates $(x, \xi)$ in $T^*(X)$ the Hamiltonian vector is given by

$$H_p = \Sigma(\partial p/\partial \xi_j \partial/\partial x_j - \partial p/\partial x_j \partial/\partial \xi_j).$$

If $q$ is another $C^1$ function on $T^*(X)$, then

$$H_p q = \; < H_p, dq > \; = \sigma(H_p, H_q) = -\sigma(H_q, H_p) = -H_q p,$$

and in local coordinates

$$H_p q = \{p, q\} = \Sigma(\partial p/\partial \xi_j \partial q/\partial x_j - \partial p/\partial x_j \partial q/\partial \xi_j).$$

$\{p, q\}$ is called the *Poisson bracket* of $p$ and $q$. For later reference we note the Jacobi identity

$$\{p, \{q, r\}\} + \{q, \{r, p\}\} + \{r, \{p, q\}\} = 0 \qquad (3.1.3)$$

or equivalently

$$H_{\{p,q\}} = H_p H_q - H_q H_p = [H_p, H_q].$$

For the proof we first observe that $[H_p, H_q]$ is a first order differential operator. This implies that (3.1.3) is independent of the second order derivatives of $r$, and similarly by the symmetry (3.1.3) is independent of the second order derivatives of $p$ and $q$. But if $p, q, r$ are all linear functions it is clear that all terms in (3.1.3) vanish so (3.1.3) must always be valid. From the Jacobi identity it follows that the (local) group of transformations defined by the vector field $H_p$ is *canonical*, that is, it preserves the symplectic form. In fact, it suffices to note that if $q_1, \ldots, q_{2n}$ are symplectic coordinates at a point $m$ and $H_p q_j$ = constant then these functions remain symplectic coordinates along the orbit of $H_p$ through $m$ since $H_p\{q_j, q_k\} = -$ $-\{q_k, \{p, q_j\}\} - \{q_j, \{q_k, p\}\} = 0$.

We now return to the Cauchy problem for (3.1.2). Let $M$ be a hypersurface in $X$ and $u_0$ a $C^1$ function with no critical point on $M$. We want to find $u$ satisfying (3.1.2) and the Cauchy boundary condition $u = u_0$ on $M$. In addition $\xi_0 = \operatorname{grad} u(x_0)$ is prescribed for some $x_0 \in M$ in such a way that $\xi_0$ restricted to $T_{x_0}(M)$ is equal to grad $u_0$. We can then extend $u_0$ to a neighborhood of $M$ so that grad $u_0 = \xi_0$ at $x_0$. If $M$ is defined by the equation $\rho = 0$ we shall then have $\text{grad}\, u = \text{grad}\, u_0 + t\, \text{grad}\, \rho$ on $M$, $t = 0$ at $x_0$, so on $M$ the equation (3.1.2) becomes $p(x, \text{grad}\, u_0 + t\, \text{grad}\, \rho) = 0$. The derivative with respect to $t$ when $t = 0$ and $x = x_0$ becomes $\{p, \rho\}(x_0, \xi_0)$. If we assume that $H_p$ (or more precisely the projection $p'_\xi\, \partial/\partial x$ of $H_p$ in $T(X)$) is *transversal* to $M$, it follows from the implicit function theorem that this equation has a unique solution in a neighborhood $V$ of $x_0$. With $u_1 = u_0 + t\rho$, the Cauchy problem is now to find a Lagrangean manifold contained in $p^{-1}(0)$ and including

$$\Lambda_0 = \{(x, \text{grad}\, u_1(x)), x \in M_0 = M \cap V\}.$$

We have already seen that $\Lambda$ must contain the integral curves of the vector field $H_p$ starting in $\Lambda_0$ and by assumption these are transversal to $\Lambda_0$. It follows that there is a unique local solution of the Cauchy problem. In fact, the local manifold generated by integral curves of $H_p$ through $\Lambda_0$ is Lagrangean at $\Lambda_0$ since $\sigma$ vanishes on $\Lambda_0$ and $\sigma(t, H_p) = \langle t, dp\rangle = 0$ if $t \in T(\Lambda_0)$. The fact that $\Lambda$ is invariant under the canonical transformations $\exp(tH_p)$ proves that $\Lambda$ is Lagrangean everywhere.

When $p$ or the Cauchy data are complex the preceding arguments break down and there is in general no solution unless $p$ and the data are analytic (see section 3.3). However we always have an analogous result for formal power series solutions at a point, and this can be applied when the data are in $C^\infty$ by considering the Taylor series expansions. We can say more if the vector field $H_p$ happens to have an integral curve $\Gamma$ with initial data $(x_0, \xi_0)$, that is, if there exists a regular $C^\infty$ curve $t \to (x(t), \xi(t)) \in T^*(X)$ with $(x(0), \xi(0)) = (x_0, \xi_0)$ and

$$0 \neq (dx/dt, d\xi/dt) = c(t)(p'_\xi, -p'_x)$$

for some complex valued function $c$. Apart from the parametrization such a curve is uniquely determined by $(x_0, \xi_0)$ since it is an integral curve of any one of the vector fields $H_{\text{Re}\,p}$ and $H_{\text{Im}\,p}$ which is $\neq 0$. If such a curve $\Gamma$ exists we can consider Taylor expansions on $\Gamma$ instead. Even if the data on $M$ are complex valued we then obtain a complex valued function $u$ such that $\text{Im}\, u$ vanishes to the second order on $\Gamma$, $\text{grad}\, \text{Re}\, u(x(t)) = \xi(t)$, the restriction of $u$ to $M$ has a given Taylor expansion at $x_0$ and $p(x, \text{grad}\, u)$ vanishes of infinite order on $\Gamma$. The last statement makes sense although $p(x, \xi)$ is not defined for complex values of $\xi$, for the derivatives of $p(x, \text{grad}\, u)$ can still be computed formally on $\Gamma$.

For a more complete though less geometrical treatment of the topics discussed here we refer to Carathéodory [1]. Since for us the equation $p = 0$

is the characteristic equation of the operator $P$, we shall use the terminology *bicharacteristic strip* (resp. *curve*) for an integral curve of the Hamiltonian field $H_p$ contained in $p^{-1}(0)$ (resp. the projection of such a curve in $X$). Note that whereas the bicharacteristic strip is non-degenerate or reduced to a point, the bicharacteristic curve may have a cusp. A simple classical example of this is given by the Tricomi equation for which $p(x, \xi) = x_2\xi_1^2 + \xi_2^2$. With suitable normalization of the parameter the bicharacteristic strips are given by $x_1 = x_1^0 - 2(ct)^3/3$, $x_2 = -t^2c^2$, $\xi_1 = c \neq 0$, $\xi_2 = -tc^2$. The cusps of the bicharacteristic curves occur when $t = 0$. (Some authors use the term bicharacteristic strip for any integral curve of $H_p$ and null bicharacteristic strip for those on which $p$ vanishes.)

### 3.2. *Operators with real principal part*

Let $P$ be a properly supported pseudo-differential operator of order $\mu$ in a manifold $X$ and assume that $P$ has a real and homogeneous principal part $p$ satisfying (3.1.1). In this case rather complete results on the propagation of singularities and existence theorems in $\mathscr{D}'/C^\infty$ have been obtained by Duistermaat and Hörmander [1]. Complete proofs of all statements in this section are given there. The following result should be compared with Theorem 1.6.5.

THEOREM 3.2.1. *If $u \in \mathscr{D}'(X)$ and $Pu = f$ it follows that $WF(u)\setminus WF(f)$ is a subset of $p^{-1}(0)$ which is invariant under the flow defined by the Hamilton vector field $H_p$ in $p^{-1}(0)\setminus WF(f)$.*

*Proof.* That $WF(u)\setminus WF(f) \subset p^{-1}(0)$ is precisely the second part of (2.2.2). To prove the other part of the theorem we consider a point $m \in WF(u)\setminus WF(f)$. If $H_p(m)$ has the radial direction the bicharacteristic curve through $m$ is a ray, and since $WF(u)$ is conic there is nothing to prove then. Otherwise we can apply

LEMMA 3.2.2. *Let $m \in T^*(X)\setminus 0$, $p(m) = 0$, and assume that $H_p(m)$ does not have the radial direction. Then there exist Fourier integral operators $A \in I^{\mu_1}(X \times \mathbf{R}^n, \Gamma')$, $B \in I^{\mu_2}(\mathbf{R}^n \times X, (\Gamma^{-1})')$ such that*

(i) *$\Gamma$ is a closed conic subset of the graph of a homogeneous canonical transformation $\chi$ from a conic neighborhood $U$ of $m$ onto a conic neighborhood $V$ of a point $\chi(m) \in T^*(\mathbf{R}^n)\setminus 0$.*

(ii) $(m, \chi(m))$ *and* $(\chi(m), m)$ *are non-characteristic points for* $A$ *and* $B$ *respectively.*

(iii) $\mu_1$ *and* $\mu_2$ *are given numbers with* $\mu_2 + \mu + \mu_1 = 1$, *and the full symbol of the pseudo-differential operator* $BPA$ *is equal to* $\xi_n$ *on a conic neighborhood of* $\chi(m)$.

*Proof.* Multiplication of $P$ by an elliptic pseudo-differential operator of order $1 - \mu$ reduces the proof to the case $\mu = 1$. The hypothesis on $H_p(m)$ then makes it possible to introduce a system of canonical coordinates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ near $m$ in $T^*(X)$, which are homogeneous of degree 0 and 1 respectively, so that $\xi_n = p$. This gives the canonical transformation $\chi$. Choosing $B$ and $A$ with reciprocal principal symbols we obtain that $BPA$ has the principal symbol $\xi_n$ near $\chi(m)$. By successive choice of the terms of decreasing order in the symbols of $B$ and $A$ one can make the lower order terms in $BPA$ vanish near $\chi(m)$.

*End of proof of Theorem* 3.2.1. With the notations of the lemma we also choose $B_1 \in I^{-\mu_1}(\mathbf{R}^n \times X, (\Gamma^{-1})')$ so that $m \notin WF(AB_1 - I)$. Then $v = B_1 u \in \mathscr{D}'(\mathbf{R}^n)$ and $\chi(m) \in WF(v)$ for against our assumption we would otherwise obtain $m \notin WF(u)$ since $u = (I - AB_1)u + Av$. (Here we are using Theorems 2.2.8 and 2.2.9.) Since

$$D_n v = (D_n - BPA)v + BP(AB_1 - I)u + BPu$$

we have $\chi(m) \notin WF(D_n v)$. Thus we have reduced the proof to the case of the operator $D_n$ for which it follows by writing down a solution of the equation $D_n v = f$ explicitly.

*Remark.* Using only pseudo-differential operators, we shall prove a more general result in section 3.5 (see also Hörmander [13]).

In the opposite direction we have

THEOREM 3.2.3. *Let* $I \subset \mathbf{R}$ *be an open interval and* $\gamma : I \to T^*(X)\backslash 0$ *be a map defining a bicharacteristic strip for* $P$ *which is injective even after composition with the projection to* $S^*(X)$. *Denote by* $\Gamma$ *the closed conic hull of* $\gamma(I)$ *and by* $\Gamma'$ *the limit points, that is, the intersection of the closed conic hull of* $\gamma(I\backslash I_0)$ *when* $I_0$ *runs over all compact intervals contained in* $I$. *For any* $\nu = 0, 1, 2, \ldots$ *one can then find* $u \in C^\nu(X)$ *such that* $WF(u)\backslash\Gamma' = \Gamma\backslash\Gamma'$ *and* $WF(Pu) \subset \Gamma'$.

Note that $\Gamma'$ is empty precisely when $\gamma$ defines a proper map from $I$ into $X$. Then we have $Pu \in C^\infty(X)$ and $WF(u) = \Gamma$.

*Proof.* We shall just indicate a slightly weaker construction for a compact subinterval $I_0$ of $I$, but the passage to the statement above is quite easy from there. Assume for example that $0 \in I_0$. There is nothing to prove if $H_p(\gamma(0))$ has the radial direction so we exclude this case. We can then choose a $n-1$ dimensional conic submanifold $N_0$ of $N = p^{-1}(0)$ through $\gamma(0)$ such that $H_p(\gamma(0))$ is not a tangent of $N_0$ and the symplectic form vanishes in $N_0$. If $\Phi(t, n)$ denotes the solution of the Hamilton-Jacobi equations $d(x, \xi)/dt = H_p(x, \xi)$ at time $t$ which is $n$ at time 0, then it is clear that there is a closed conic neighborhood $V_0 \subset N_0$ of $\gamma(0)$ such that the map

$$I_0 \times V_0 \ni (t, n) \to \Phi(t, n)$$

is injective. Hence it defines a closed conic subset $\Lambda$ of a Lagrangean manifold on which $p = 0$. (See the discussion of the Cauchy problem in section 3.1.) One can now choose $u \in I^k(X, \Lambda)$ with $WF(Pu)$ close to $\Phi((\partial I_0) \times V_0)$ so that the principal symbol of $u$ has a given restriction to $N_0$ with support in a small conic neighborhood of $\gamma(0)$ in $N_0$. The crucial point is that for phase functions $\varphi$ defining $\Lambda$ locally we have $p(x, \varphi'_x) = 0$ when $\varphi'_\theta = 0$. From this one concludes that to make the principal symbol of $Pu$ vanish except at $\Phi(\partial I_0 \times V_0)$ means to solve differential equations along the bicharacteristics of $P$ contained in $\Lambda$. (See the remarks on geometrical optics in the introduction.) As usual one can then successively determine terms of decreasing order in the symbol of $u$ so that the symbol of $Pu$ is of order $-\infty$ except at $\Phi((\partial I_0) \times V_0)$. If the order $k$ is suitably chosen the desired properties are obtained. (To see that $WF(u)$ can be squeezed into $\Gamma$ and not only a neighborhood one can either use functional analysis (see section 3 in Hörmander [7]) or more general symbols. A third possibility is indicated in the proof of Theorem 3.4.1 below.)

*Remark.* Zerner [1] and Hörmander [7] have given similar results which are weaker in that they are local and that they require $H_p$ not to be a tangent to the fiber of $T^*(X)$ so that the bicharacteristic curve is regular. These constructions do not require the global definition of Fourier integral operators as the proof of Theorem 3.2.3 does.

We can now give an analogue of part iii) of Theorem 1.4.6.

THEOREM 3.2.4. *Assume that no complete bicharacteristic strip of P stays over a compact set in X. Then the following conditions are equivalent:*

a) *P defines a surjective map from* $\mathscr{D}'(X)$ *to* $\mathscr{D}'(X)/C^\infty(X)$.

b) *For every compact set $K \subset X$ there is another compact set $K' \subset X$, which can be taken empty when $K$ is empty, such that $u \in \mathscr{E}'(X)$ and* sing supp ${}^tPu \subset K$ *implies* sing supp $u \subset K'$.

c) *For every compact set $K \subset X$ there is another compact set $K' \subset X$ such that any interval on a bicharacteristic curve with respect to $P$ having endpoints in $K$ must belong to $K'$.*

*Proof.* b) implies a) by Theorem 1.2.4. Assume that c) is fulfilled, and let $u \in \mathscr{E}'(X)$. If $m \in WF(u) \setminus WF(Pu)$ it follows from Theorem 3.2.1 that each bicharacteristic half strip through $m$ must contain some point in $WF(Pu)$ unless it stays in $WF(u)$ and therefore over a compact set. However, if a bicharacteristic half strip stays over a compact set, then the bicharacteristic strip through any one of its limit points in the sphere bundle stays over this compact set in both directions which we have excluded by hypothesis. Hence $m$ lies on an interval of a bicharacteristic strip with end points over $K$ which proves that b) follows from c). By using a more precise version of Theorems 3.2.1 and 3.2.3 and an argument close to the proof of Theorem 3.6.3 in Hörmander [1] one shows that a) implies c).

Assuming still that $P$ has no bicharacteristic strip which stays over a compact set in $X$, we set $N = p^{-1}(0) \subset T^*(X) \setminus 0$ and let $C \subset N \times N$ be the *bicharacteristic relation* of pairs of points in $N$ which are on the same bicharacteristic strip. It is then easy to verify that $C$ is a homogeneous canonical relation if and only if condition c) in Theorem 3.2.4 is fulfilled. Let $C^+$ (resp. $C^-$) be the subset of pairs $(n_1, n_2)$ with $n_1$ on the forward (backward) bicharacteristic strip starting at $n_2$. Using the calculus of Fourier integral operators outlined in section 2.3 and Lemma 3.2.2 above one can prove

THEOREM 3.2.5. *Assume that no complete bicharacteristic strip of $P$ stays over a compact set in $X$ and that condition* c) *in Theorem* 3.2.4 *is fulfilled. Then there exist right parametrices $E^+$ and $E^-$ for $P$, that is, operators such that $PE^+ - I$ and $PE^- - I$ have $C^\infty$ kernels, with the following properties*[1]*:*

a) *$E^\pm$ are continuous linear maps from $H_{(s)} \cap \mathscr{E}'$ to $H_{(s+\mu-1)}$ for every $s$.*

[1]) (Added in proof) In fact $E^\pm$ are also left parametrices and $E^\pm E^- \in I^{1/2-\mu}(X \times X, C')$. (See Duistermaat - Hörmander [1])

b) *$WF'(E^+)$ (resp. $WF'(E^-)$) is contained in $\Delta^* \cup C^+$ (resp. $\Delta^* \cup C^-$) where $\Delta^*$ is the diagonal in $T^*(X)\backslash 0 \times T^*(X)\backslash 0$.*

c) *Outside $\Delta^*$ the kernels of $E^+$ and $E^-$ are in $I^{1/2-\mu}(X \times X, C')$.*

*Condition* b) *determines $E^\pm$ uniquely* mod $C^\infty$.

A still better result, essentially due to Grušin [1] for operators with constant coefficients, can be obtained in the following way. Let $A^+$ and $A^-$ be properly supported pseudo-differential operators with $A^+ + A^- = I$. With $E^+$ and $E^-$ as in Theorem 5.3.7 we obtain a new parametrix $E$ if we set

$$E = E^+A^+ + E^-A^- .$$

It will inherit the continuity properties of $E^+$ and $E^-$ listed above, and

$$WF'(E) \subset \Delta^* \cup \{(m, n) \in C^\pm, n \in WF(A^\pm)\} .$$

Using operators with symbols satisfying (2.1.3)′ one can arrange that $WF(A^\pm) = F^\pm$ are any closed cones in $T^*(X)\backslash 0$ with union equal to $T^*(X)\backslash 0$. By condition c) in Theorem 3.2.4 one obtains for a suitable choice of $F^+$ and $F^-$ a parametrix which can be extended to a continuous map from $H_{(s)}(X)$ to $H_{(s+\mu-1)}(X)$ for every $s$. This gives back part a) of Theorem 3.2.4 in a more constructive way.

We have only given global existence theorems here. However, local results follow immediately and they require only that no bicharacteristic strip for $P$ stays forever over a fixed point in $X$. In the next section we shall discuss some more serious obstacles to local solvability which may occur when $p$ is complex valued.

### 3.3. *Necessary conditions for local solvability and hypoellipticity*

We shall now allow the principal part $p$ of the pseudo-differential operator $P$ to be complex valued. That this leads to a drastic change of the situation discussed in section 3.2 was first realized by H. Lewy [1]. He found that the equation

$$(\partial/\partial x_1 + i\partial/\partial x_2 + 2i(x_1 + ix_2)\,\partial/\partial x_3)\,u = f$$

does not have a solution in any open set for suitably chosen $f \in C^\infty(\mathbf{R}^3)$. Starting from this example some necessary and some sufficient conditions for existence of (local) solutions were given by the author (see Hörmander [1, Chap. VI, VIII] and for the case of pseudo-differential operators Hörmander [3]). Mizohata [1] observed that for the equation

$$(\partial/\partial x_1 + i x_1^k \partial/\partial x_2)\, u = f \tag{3.3.1}$$

there is an existence theorem for even $k$ but no solutions near the $x_2$ axis for suitable $f$ if $k$ is odd. With this example as starting point more precise conditions for local existence of solutions have been obtained by Nirenberg and Trèves [1], [2] (see also Trèves [2], [3]) and by Egorov [2], [3]. We shall discuss these results here in a somewhat more precise form made possible by the notion of wave front sets.

DEFINITION 3.3.1. The operator $P$ is said to be solvable at $x_0 \in X$ if there is an open neighborhood $V$ of $x_0$ such that for every $f \in C^\infty(X)$ one can find $u \in \mathscr{D}'(X)$ with $Pu = f$ in $V$.

Introducing a positive $C^\infty$ density in $X$ we can form the adjoint ${}^tP$ of $P$ and write the equation $Pu = f$ in $V$ as

$$< u, {}^tPv > \;=\; < f, v > , \quad v \in C_0^\infty(V) .$$

We may assume that $V \subset\subset X$. Solvability implies that the bilinear form

$$C^\infty(X) \times C_0^\infty(V) \ni (f, v) \to \; < f, v >$$

is separately continuous if for $f$ we take the $C^\infty$ topology and for $v$ the weakest topology which makes the mapping $v \to {}^tPv \in C^\infty(X)$ continuous. Hence the form is continuous (Banach-Steinhaus), which means that for some semi-norms $N_1$, $N_2$ in $C^\infty(X)$

$$| < f, v > | \leqq C N_1(f)\, N_2({}^tPv), \; f \in C^\infty(X), v \in C_0^\infty(V) .$$

$N_1$ and $N_2$ are continuous semi-norms in $C^k(X)$ for some $k$. The estimate is clearly valid also for $f \in C^k(X)$, and an application of the Hahn-Banach theorem to the map

$${}^tPv \to \; < f, v >$$

shows that for every $f \in C^k(X)$ one can find $u \in \mathscr{E}'^k(X)$ so that $Pu = f$ in $V$. We have therefore proved

PROPOSITION 3.3.2. *If $P$ is solvable at $x_0$, then there is a neighborhood $V$ of $x_0$ and an integer $k$ such that for every $f \in C^k(X)$ one can find $u \in \mathscr{E}'^k(X)$ with $Pu = f$ in $V$.*

To prove that $P$ is not solvable at $x_0$ it is therefore sufficient to exhibit arbitrarily smooth functions $f$ such that $Pu - f$ is not smooth near $x_0$ for any distribution $u$. This property has the advantage that it can be localized in the cotangent bundle as indicated in section 2.2:

DEFINITION 3.3.3. *If* $(x_0, \xi_0) \in T^*(X) \backslash 0$ *and* $f \in \mathcal{D}'(X)$, *we shall say that* $f \in P\mathcal{D}'(X)$ *at* $(x_0, \xi_0)$ *if one can find* $u \in \mathcal{D}'(X)$ *so that* $(x_0, \xi_0) \notin WF(Pu-f)$. *We shall say that P is solvable at* $(x_0, \xi_0)$ *if this is possible for every f.*

Solvability of $P$ at a point $(x_0, \xi_0) \in T^*(X) \backslash 0$ is closely related to smoothness there of solutions of the adjoint equation $P^*u = f$ when $f$ is smooth and $WF(u)$ is close to $(x_0, \xi_0)$. Such existence and smoothness questions will therefore be studied simultaneously in what follows. To trace the origin of our arguments we first digress to discuss boundary problems for elliptic operators briefly.

Consider as an example the Laplace equation $\Delta u = 0$ in an open set $X \subset \mathbf{R}^n$ with a differential boundary condition $Bu = f$ on the smooth boundary $\partial X$. If $u_0$ is the restriction of $u$ to $\partial X$, then $u$ is the Poisson integral of $u_0$ and the boundary condition $Bu = f$ can be written as a pseudo-differential equation $\tilde{B}u_0 = f$ where the principal symbol of $\tilde{B}$ is easy to compute. In this way the study of *elliptic* boundary problems (see Agmon-Douglis-Nirenberg [1] or Hörmander [1, Chap. X]) can always be reduced to the study of an *elliptic* system of pseudo-differential operators on the compact manifold $\partial X$. The reduction is possible quite generally, however. In particular we can take $B = \partial/\partial v$ where $v$ is a non-vanishing vector field on $\partial X$ such that the equation $\langle v, N \rangle = 0$ defines a non-singular submanifold $Y$ of $\partial X$, if $N$ is the interior normal of $\partial X$. From the results related to Lewy's equation referred to above it follows that there is (local) solvability of the boundary problem if on $Y$ the derivative of $\langle v, N \rangle$ in the direction $v$ (which is tangential to $\partial X$ on $Y$) is negative whereas there is a non-existence theorem if it is positive. For regularity of solutions the opposite signs are required. (See Borelli [1], Hörmander [3].) This strange result was explained by Egorov and Kondrat'ev [1] who found that in the two cases one should respectively introduce an additional boundary condition on $Y$ or allow a discontinuity there. The problem then becomes well posed and solutions are smooth apart from a smooth jump. The proof of Egorov and Kondrat'ev attacked the boundary problem directly but their result can be translated to a property of a certain pseudo-differential operator which is elliptic outside a submanifold $Y$ of codimension one. General theorems of this type have been proved by Eškin [1] and Sjöstrand [1].

Here we shall to a large extent follow Sjöstrand but will only deal with the situation corresponding to $\partial u/\partial v = 0$ and given restriction of $u$ to $Y$.

Let us first consider the typical example given by equation (3.3.1) with $f = 0$. If it is possible to take Fourier transforms with respect to $x_2$, the equation becomes

$$(\partial/\partial x_1 - x_1{}^k \xi_2)\hat{u}(x_1, \xi_2) = 0$$

with the solution $\hat{u}(x_1, \xi_2) = C(\xi_2) \exp(\xi_2 x_1{}^{k+1}/(k+1))$. If $k$ is *odd* we set for $v \in C_0^\infty(\mathbf{R})$

$$Ev(x) = (2\pi)^{-1} \int_{-\infty}^{0} \exp\left(\xi_2(ix_2 + x_1{}^{k+1}/(k+1))\right) v(\xi_2)\, d\xi_2 =$$
$$= (2\pi)^{-1} \iint_{\xi_2 < 0} \exp\left(\xi_2(i(x_2 - y) + x_1{}^{k+1}/(k+1))\right) v(y)\, dy d\xi_2 .$$

From the results of section 2.3 it follows that $E$ maps $C_0^\infty(\mathbf{R})$ to $C^\infty(\mathbf{R}^2)$ and $\mathscr{E}'(\mathbf{R})$ to $\mathscr{D}'(\mathbf{R}^2)$ continuously, and it is clear that $PEv = 0$ if $P = (\partial/\partial x_1 + ix_1{}^k \partial/\partial x_2)$. Let $\gamma : \mathbf{R} \ni x_2 \to (0, x_2)$ be the inclusion of the $x_2$-axis. Since the $x_2$-axis is non-characteristic with respect to $P$, it follows from (2.2.2) and Theorem 2.2.5 that the restriction $\gamma^* Ev(x_2)$ is defined, and clearly we have

$$\gamma^* Ev(x_2) = (2\pi)^{-1} \int_{-\infty}^{0} e^{ix_2\xi_2} \hat{v}(\xi_2)\, d\xi_2 .$$

Using Theorem 2.3.1 we see that

$$WF'(E) = \{(x_1, \xi_1, x_2, \xi_2, y_2, \eta_2);\ x_1 = \xi_1 = 0,\ x_2 = y_2,\ \xi_2 = \eta_2 < 0\}.$$

For suitable choice of $v$ we obtain a solution $u = Ev$ of $Pu = 0$ with $WF(u)$ equal to any closed subset of $F = \{(x_1, \xi_1, x_2, \xi_2);\ x_1 = \xi_1 = 0,\ \xi_2 < 0\}$ and conclude that $P$ is not hypoelliptic. Moreover, if $u \in \mathscr{E}'$ and $P^*u = f$, then $E^*f = 0$ because $E^*P^* = (PE)^* = 0$. In case we only have $(x_0, \xi_0) \notin WF(P^*u - f)$ for some $u \in \mathscr{D}'$ we can still conclude that $WF'(E^*)\ (x_0, \xi_0) \notin WF(E^*f)$. For every point in $F$ this is a non-trivial necessary condition in order that $f \in P\,\mathscr{D}'(X)$ at $(x_0, \xi_0)$. (By studying the inhomogeneous equation $Pu = f$ Sjöstrand also obtains the sufficiency.)

Let us more generally consider a pseudo-differential operator such that the principal symbol in a local coordinate system with coordinates varying over $\mathbf{R}^n$ is of the form

$$p(x, \xi) = \xi_n + ix_n{}^k q(x, \xi) \tag{3.3.2}$$

when $\xi$ is in a conic neighborhood of $\xi_0 = (\theta_0, 0) \neq 0$ and $x$ is near $0 \in \mathbf{R}^n$.

PROPOSITION 3.3.4. *Let $p$ be of the form* (3.3.2) *with $k$ odd and* $\operatorname{Re} q(0, \xi_0) < 0$. *If $B$ is a pseudo-differential operator in* $\mathbf{R}^{n-1}$ *with* $WF(B)$ *contained in a sufficiently small conic neighborhood of* $(0, \theta_0)$, *there exists a Fourier integral operator* $E : C_0^\infty(\mathbf{R}^{n-1}) \to C_0^\infty(\mathbf{R}^n)$ *with continuous extension from* $\mathscr{E}'(\mathbf{R}^{n-1})$ *to* $\mathscr{E}'(\mathbf{R}^n)$ *such that*

(i) *$PE$ has a $C^\infty$ kernel.*

(ii) $WF'(E) = \{(x', x_n, \xi', \xi_n; y', \eta'), x_n = \xi_n = 0,$
$(x', \xi') = (y', \eta') \in WF(B)\}$

(iii) $\gamma^* E = B$ *if* $\gamma(x') = (x', 0) \in \mathbf{R}^n, x' \in \mathbf{R}^{n-1}$

*Proof.* Let $b$ be a symbol for $B$ vanishing outside a small conic neighborhood of $(0, \theta_0)$. In order to have (iii) we wish to write $E$ in the form

$$Ev(x) = (2\pi)^{1-n} \int e^{i\varphi(x,\theta)} a(x, \theta) \hat{v}(\theta) d\theta = \tag{3.3.3}$$
$$= (2\pi)^{1-n} \iint e^{i(\varphi(x,\theta) - <y',\theta>)} a(x, \theta) v(y') dy' d\theta$$

where

$$\varphi(x, \theta) = < x', \theta >, a(x, \theta) = b(x', \theta) \text{ when } x_n = 0 . \tag{3.3.4}$$

In order to obtain (i) the rules of geometrical optics require that one first solves the characteristic equation

$$\partial\varphi/\partial x_n + i x_n^k q(x, \partial\varphi/\partial x) = 0 \tag{3.3.5}$$

approximately with the initial data of (3.3.4). By the general remarks made in section 3.1 or directly by just computing what $\partial^j \varphi/\partial x_n^j$ must be when $x_n = 0$ for every $j$, we obtain a solution $\varphi$ of infinite order when $x_n = 0$, and

$$\varphi(x, \theta) = < x', \theta > - i x_n^{k+1} q(x', 0, \theta, 0)/(k+1) + O(x_n^{k+2}) .$$

Note that, in a neighborhood of $(0, \theta_0)$ in which the support of $a$ will lie,

$$\operatorname{Im} \varphi(x, \theta) \geqq c\, x_n^{k+1} | \theta | \tag{3.3.6}$$

for some $c > 0$, which gives (ii) in view of Theorem 2.3.1. Following the rules of geometrical optics (see also the parametrix construction in section 2.1) we determine successively the terms in an asymptotic series for $a$ such that (i) is fulfilled. In doing so we can let $P$ act under the integral sign in (3.3.3) and use the same formal expansion of $P(e^{i\varphi(x,\theta)} a(x, \theta))$ as if $P$ were a differential operator (cf. Hörmander [3], Nirenberg-Trèves [2] and Hörmander [4, Theorem 2.6]).

We can now continue the argument precisely as in the example above. It follows that we can choose $u$ with $Pu \in C^\infty$ and $WF(u)$ equal to any closed cone $F$ in a sufficiently small neighborhood of $(0, \xi_0)$ in $p^{-1}(0)$. We can also choose $f$ as smooth as we please so that $f$ is not in $P^*\mathscr{D}'(X)$ at any point in $F$. Putting this conclusion in a form which is invariant under the equivalence used in Lemma 3.2.2 we shall obtain the main results of this section.

PROPOSITION 3.3.5. *Let $N_j$, $j = 1, 2, 3$, be the sets of all $m \in T^*(X)\backslash 0$ with $p(m) = 0$ having the following properties*

($N_1$) *There exist Fourier integral operators $A$, $B$ with the properties* (i), (ii) *in Lemma* 3.2.2 *such that the principal symbol of BPA satisfies the conditions in Proposition* 3.3.4 *at $\chi(m)$.*

($N_2$) $H^I\{p_1, p_2\}(m) = 0$, $|I| < \mu$; $H^I\{p_1, p_2\}(m) = \lambda^I c$, $|I| = \mu$, *for some even integer $\mu \geqq 0$, $(\lambda_1, \lambda_2) \in \mathbf{R}^2\backslash 0$, and real $c < 0$; here we have written $p = p_1 + ip_2$, denoted by $H^I$ any product of $|I|$ Hamiltonian first order operators $H_{p_1}$ or $H_{p_2}$ and by $\lambda^I$ the corresponding product of $\lambda_1$ or $\lambda_2$. If $\mu \neq 0$ then $\lambda_2 H_{p_1}(m) - \lambda_1 H_{p_2}(m) = 0$.*

($N_3$) *For some even integer $\mu \geqq 0$ and complex number $z$ we have*

$$(\operatorname{Re} zH_p)^j\{\bar{p}, p\}(m)/2i = 0 \text{ for } j < \mu \text{ and } < 0 \text{ for } j = \mu\,.$$

*Then the closures of $N_1$, $N_2$, $N_3$ in $T^*(X)\backslash 0$ are equal.*

*Proof.* $N_1 \subset N_2$. Since ($N_2$) is invariant under canonical transformations and multiplication of $p$ by a non-vanishing factor $q$ (or even transformation of $(p_1, p_2)$ by a matrix with positive determinant) it suffices to check ($N_2$) when $p_1(x, \xi) = \xi_n - x_n{}^k \operatorname{Im} q(x, \xi)$, $p_2(x, \xi) = x_n{}^k \operatorname{Re} q(x, \xi)$ and $\operatorname{Re} q < 0$. Then we have $\{p_1, p_2\} = kx_n{}^{k-1} \operatorname{Re} q(x, \xi) + O(x_n{}^k)$, $H_{p_1} - \partial/\partial x_n$ and $H_{p_2}$ vanish when $x_n = 0$ if $k > 1$. Since $x_n = 0$ we obtain ($N_2$) with $\mu = k - 1$, $c = k!\operatorname{Re} q$ and $\lambda = (1, 0)$ if $k > 1$. That $N_2 \subset N_3$ is trivial. To show that $N_3$ is in the closure of $N_1$ it suffices to consider a point in $N_3$ such that $z = 1$, that is,

$$H_{p_1}{}^j p_2(m) = 0 \text{ for } j \leqq \mu, H_{p_1}{}^{\mu+1} p_2(m) < 0\,.$$

Since $p_2(m) = 0$ it follows that $H_{p_1}(m)$ does not have the radial direction. According to Lemma 3.2.2 we can therefore choose Fourier integral operators $A$ and $B$ satisfying conditions (i), (ii) there so that the principal part of $BA$ is real and the real part of the principal symbol of $BPA$ is $\xi_n$ near $\chi(m)$.

To economize notation we assume that already $p_1(\xi) = \xi_n$. Then $H_{p_1} = \partial/\partial x_n$ and our hypotheses are now that $m = (0; \theta_0, 0)$, $\partial^j/\partial x_n{}^j p_2(0; \theta_0, 0) = 0$ for $j \leqq \mu$ and $< 0$ for $j = \mu + 1$. Hence $p_2(0, x_n; \theta_0, 0)$ has the sign of $-x_n$ for small $x_n$. It follows that the equation $p_2(x', x_n; \xi', 0) = 0$ for $(x', \xi')$ close to $(0, \theta_0)$ has at least one zero where $p_2$ for increasing $x_n$ changes sign from plus to minus. If we choose such a zero close to $m$ of minimum multiplicity $k$, necessarily odd, we may conclude from the implicit function theorem applied to $\partial^{k-1} p_2/\partial x_n{}^{k-1}$ that the zeros of $p$ nearby are defined by $\xi_n = 0$ and an equation $x_n = r(x', \xi')$ with $r \in C^\infty$ homogeneous of degree 0 with respect to $\xi'$. Noting that the Poisson bracket $\{\xi_n, x_n - r(x', \xi')\}$ is 1 it is easy to add further canonical coordinates to $\xi_n$ and $x_n - r(x', \xi')$ to obtain a homogeneous canonical transformation changing these functions to $\xi_n$ and $x_n$. Implementing this by Fourier integral operators as in Lemma 3.2.2 again we see that at some point corresponding to a point arbitrarily close to $m$ the transformed operator $BPA$ will have a principal part of the form $\xi_n + iq_1$ where $q_1(x, \xi', 0) = x_n{}^k q(x, \xi')$ and $q < 0$. Thus the principal part can be written $\xi_n(1 + is) + ix_n{}^k q$ where $s$ is real. Multiplication by an elliptic operator with symbol $(1 + is)^{-1}$ reduces it to the desired form and completes the proof.

Definition 3.3.6. The closure of any one of the sets $N_1$, $N_2$, $N_3$ in Proposition 3.3.5 will be denoted by $N_-(p)$, and we write $N_+(p) = N_-(\bar{p})$ which corresponds to changing the signs in the definition of $N_1$, $N_2$, $N_3$.

Note that in the case of differential operators the fact that $p(x, \xi) = (-1)^\mu p(x, -\xi)$ implies that $N_+(p)$ and $N_-(p)$ differ by multiplication with $-1$ in the fibers of $T^*(X)$. Thus they are simultaneously empty. This is not the case for pseudo-differential operators. For example, the study of the oblique derivative problem mentioned above leads to

$$p(x, \xi) = \xi_n + icx_n |\xi|$$

where $c \in \mathbf{R}\backslash 0$. Then $p = 0$ is equivalent to $x_n = \xi_n = 0$ and $\{Re\, p, Im\, p\} = c|\xi|$ has the sign of $c$ there, so either $N_+$ or $N_-$ is empty but not both.

From Propositions 3.3.4 and 3.3.5 we obtain by simple functional analysis:

Theorem 3.3.7. *Let $F_+$ and $F_-$ be arbitrary closed cones contained in $N_+(p)$ and $N_-(p)$. For every $k > 0$ one can find $f \in C^k(X)$ with*

*$WF(f) = F_+$ such that $f$ is not in $P\mathcal{D}'(X)$ at any point in $F_+$. One can also find $u \in \mathcal{D}'(X)$ with $WF(u) = F_-$ and $Pu \in C^\infty(X)$.*

The theorem shows that every (local) existence theorem must assume that $N_+(p) = \varnothing$ and that hypoellipticity requires that $N_-(p) = \varnothing$. The first statement is the necessary condition of Egorov, Nirenberg and Trèves referred to above.

In the notation of Proposition 3.3.5 Egorov's form of the condition $N_-(p) = \varnothing$ is $N_3 = \varnothing$. To arrive at the version of Nirenberg and Trèves we consider a point $m \in T^*(X)\backslash 0$ with $p(m) = 0$ and $d\,\mathrm{Re}\,p(m) \neq 0$. The equation $\mathrm{Re}\,p = 0$ defines a smooth hypersurface $S$ containing $m$, and through each point in $S$ there is an oriented integral curve of $H_{\mathrm{Re}\,p}$ which stays in $S$. Since in condition $(N_2)$ we must have $\lambda_1 \neq 0$ if $\mu > 0$, it follows from $(N_2)$ and $(N_3)$ that $N_-(p) = \varnothing$ if and only if in a neighborhood of $m$ in $S$ the restriction of $\mathrm{Im}\,p$ to integral curves of $H_{\mathrm{Re}\,p}$ never has a zero of finite order where the sign changes from positive to negative. This is the condition of Nirenberg and Trèves. They conjectured that a necessary and sufficient condition for solvability at $m$ of the adjoint (if $H_p$ does not have the radial direction) is that such sign changes do not occur at any zeros (of finite or infinite order). A proof of the invariance of this condition under multiplication of $p$ by a non-vanishing factor was given in Nirenberg-Trèves [2, appendix]. In fact, they discuss a semiglobal version of the same condition but the statements are not precise in this respect. Note that solvability of $P$ at $(x_0, \xi_0)$ for every $\xi_0 \neq 0$ does not imply solvability at $x_0$. An example is the differential operator in $\mathbf{R}^2$

$$P = x_1\, \partial/\partial x_2 - x_2\, \partial/\partial x_1$$

which in view of Lemma 3.2.2 is locally solvable at any point $(x_0, \xi_0)$ but is obviously not solvable at 0. In Theorem 3.2.4 such behavior is ruled out by the assumption that bicharacteristic curves cannot lie in a compact set and similar conditions should be imposed in general.

## 3.4. *Further necessary conditions for hypoellipticity*

The standard definition of hypoellipticity which we have used throughout is that $P$ is hypoelliptic if

$$\text{sing supp}\, u = \text{sing supp}\, Pu, \quad u \in \mathcal{D}'(X). \tag{3.4.1}$$

This means that for every open set $Y \subset X$

(3.4.2) $$u \in \mathscr{D}'(X), Pu \in C^\infty(Y) \Rightarrow u \in C^\infty(Y).$$

For operators with variable coefficients this condition may be fulfilled for a fixed $Y$, for example $Y = X$, while (3.4.1) is not valid. For example, if $X = \{x \in \mathbf{R}^2, 1 < |x| < 2\}$ and $P = x_1\partial/\partial x_2 - x_2\partial/\partial x_1 + 1$ then $P$ is not hypoelliptic but (3.4.2) is valid if $Y = X$. On the other hand, using the notion of wave front sets we can also consider a stronger property than (3.4.1)

(3.4.3) $$WF(u) = WF(Pu), u \in \mathscr{D}'(X).$$

Such operators will be called *strictly hypoelliptic* here. All hypoelliptic differential operators with constant coefficients as well as the hypoelliptic operators discussed in Hörmander [4] (see section 2.1) are strictly hypoelliptic. (It seems quite clear that if wave front sets had been considered some 15 to 20 years ago, then (3.4.3) rather than (3.4.1) would have been taken as definition of hypoelliptic operators.)

An operator $P \in L^\mu(X)$ is called *subelliptic* if for some $\delta > 0$ and real $s$

(3.4.4) $$u \in H_{(s)}(X) \cap \mathscr{E}'(X), Pu \in H_{(s+1-\mu)}(X) \Rightarrow u \in H_{(s+\delta)}(X).$$

Elliptic operators correspond to $\delta = 1$. From (3.4.4) it follows that we have a seemingly much stronger property: For any $t \in \mathbf{R}$

(3.4.5) $$u \in \mathscr{D}'(X), Pu \in H_{(t)} \text{ at } m \in T^*(X)\backslash 0 \Rightarrow u \in H_{(t+\mu-1+\delta)} \text{ at } m.$$

In particular, *subellipticity implies strict hypoellipticity*. To prove (3.4.5) we choose a real number $r$ so that $u \in H_{(r)}$ at $m$. Assuming that $r \leqq t + \mu - 1$ we shall prove that $u \in H_{(r+\delta)}$ at $m$; by iteration this gives (3.4.5). Choose a pseudo-differential operator $A$ of order $r - s$ which is non-characteristic at $m$ so that $Au \in H_{(s)}(X) \cap \mathscr{E}'(X)$ and $APu \in H_{(t-r+s)}(X)$. We have

$$PAu = APu - [A, P]u.$$

Here $APu \in H_{(t-r+s)} \subset H_{(s+1-\mu)}$ and $[A, P]$ is of order $\leqq r - s + \mu - 1$ so $[A, P]u \in H_{(s-\mu+1)}$ also. It follows from (3.4.4) that $Au \in H_{(s+\delta)}(X)$, hence that $u \in H_{(r+\delta)}$ at $m$.

Subelliptic operators were characterized by Hörmander [3] for $\delta = 1/2$ by means of a localization method which is also valid for arbitrary $\delta > 0$ (see Hörmander [4]). In a series of papers Yu. V. Egorov has analyzed the localized estimates for arbitrary $\delta > 0$; their complexity increases very much as $\delta \to 0$. In Egorov [2] it was announced that (3.4.4) (or (3.4.5)) is valid if and only if $N_-(p) = \varnothing$ (see Definition 3.3.6) and

(3.4.6) $$H_p{}^j \bar{p}(m) = 0, 0 \leqq j \leqq \mu \Rightarrow \delta \leqq 1/(\mu+2).$$

Here we have used the notations in Proposition 3.3.5 and $\mu$ may be equal to 0. However, according to the lecture by Egorov at the International Congress in Nice there is a gap in his proof of sufficiency when $H_{\mathrm{Re}\,p}$ and $H_{\mathrm{Im}\,p}$ are linearly dependent. (When they are linearly independent a proof has been given in Egorov [3] and another is easily obtained by combination of the results in Hörmander [3] and [5].)

In this section we shall derive other necessary conditions for hypo-ellipticity from constructions of solutions with small singularities. These are variants of Theorem 3.2.3. The first result is a more precise version of one due to Trèves [5], [7].

THEOREM 3.4.1. *Let $I$ be an interval $\subset \mathbf{R}$ and $I \ni t \to \gamma(t) \in T^*(X) \setminus 0$ a bicharacteristic strip for $P$, that is, $0 \neq \gamma'(t)$ is proportional to $H_p(\gamma(t))$ for every $t \in I$. If $I_0$ is a sufficiently small neighborhood of a point $t_0 \in I$ and $\Gamma$ (resp. $\Gamma'$) is the closed conic hull of $\gamma(I_0)$ (resp. $\gamma(\partial I_0)$) one can for $\nu = 0, 1, 2, \ldots$ find $u \in C^\nu(X)$ so that $WF(u) = \Gamma$, $WF(Pu) \subset \Gamma'$.*

*Proof.* There is nothing to prove if $\gamma(t)$ has a constant projection on the cosphere bundle. Otherwise we can after an application of Lemma 3.2.2 assume that $p'_\xi(\gamma(t_0)) \neq 0$. Let $\gamma(t_0) = (x_0, \xi_0)$ and choose a function $\varphi$ so that

(i) $\varphi(x) = \langle x - x_0, \xi_0 \rangle + i|x - x_0|^2$ in $\Sigma$ where $\Sigma$ is a plane in $\mathbf{R}^n$ through $x_0$ which is transversal to $p'_\xi(\gamma(t_0))$.

(ii) If $\gamma(t) = (x(t), \xi(t))$ then grad $\varphi(x(t)) = \xi(t))$ for $t$ near $t_0$, and $p(x, \operatorname{grad} \varphi) = 0$ of infinite order on the bicharacteristic curve $\{x(t)\}$.

By the remarks on first order differential equations given in section 3.1 it is possible to choose $\varphi$ locally with these properties. Since $\operatorname{Im} \varphi$ vanishes to the second order on $\{x(t)\}$ it follows from (i) that

(3.4.6) $$\operatorname{Im} \varphi(x) \geqq c\, d(x)^2$$

where $c > 0$ and $d(x)$ is the distance from $x$ to the curve. One can now repeat the proof of Theorem 3.2.3 to obtain $u$ in the form of a Fourier integral operator with phase function $\theta\varphi(x)$.

It seems difficult to improve Theorem 3.4.1 to a global result analogous to Theorem 3.2.3 as one would like to do in order to study (3.4.2) for a fixed $Y$. To do so we would first have to give a global definition of spaces of

Fourier integral operators which correspond locally to phase functions $\varphi$ such as the one just constructed. Besides the curve $\gamma(t)$, the most important data contained in $\varphi$ are the second order derivatives of $\varphi$ along the curve. Let $V(t)$ be the tangent space of $T(T^*(X))$ at $\gamma(t)$ reduced modulo $\gamma'(t)$ and restricted to the orthogonal space of $\gamma'(t)$. Then $V(t)$ is symplectic, and if $V_{\mathbf{C}}(t)$ is the complexification, the Hamiltonian field $H_p$ gives symplectic bijections $\chi_{st} : V_{\mathbf{C}}(t) \to V_{\mathbf{C}}(s)$. The Lagrangean plane defined in local coordinates by $\delta\xi = \varphi''_{xx}\,\delta x$ gives a Lagrangean plane $\lambda(t)$ in $V_{\mathbf{C}}(t)$ with $\chi_{st}\,\lambda(t) = \lambda(s)$. To have (3.4.6) we must require that $\lambda(t)$ is positive in the sense that

$$\operatorname{Im}\sigma(T, \overline{T}) > 0 \text{ if } 0 \neq T \in \lambda(t)\,.$$

This condition is preserved by symplectic transformations which preserve the real spaces $V(t)$ but not by general complex symplectic transformations. Thus positivity of $\lambda(t)$ does not imply positivity of $\lambda(s)$. This is why we could make a global statement of Theorem 3.2.3 but not of Theorem 3.4.1. However, we have no examples which prove that this global difficulty is not merely due to the method of proof.

Next we consider a point $m \in p^{-1}(0) \backslash (N_+(p) \cup N_-(p))$ where $H_{\operatorname{Re}p}(m)$ and $H_{\operatorname{Im}p}(m)$ are linearly independent. Then $p^{-1}(0)$ is near $m$ a manifold of codimension 2 on which $\{\operatorname{Re}p, \operatorname{Im}p\} = 0$; conversely, these conditions imply that $m \notin N_+(p) \cup N_-(p)$. By the Jacobi identity it follows that $[H_{\operatorname{Re}p}, H_{\operatorname{Im}p}] = H_{\{\operatorname{Re}p,\, \operatorname{Im}p\}}$ is a linear combination of $H_{\operatorname{Re}p}$ and $H_{\operatorname{Im}p}$ on $p^{-1}(0)$. In view of the Frobenius theorem we conclude that through $m$ there passes a two dimensional local integral manifold of the vector fields $H_{\operatorname{Re}p}$, $H_{\operatorname{Im}p}$, contained in $p^{-1}(0)$ of course. This we call the bicharacteristic strip through $m$. Combination of the proof of Theorem 8.3 in Hörmander [7] with an analogue of Lemma 3.2.2 gives easily

THEOREM 3.4.2. *Let $m \in p^{-1}(0) \backslash (N_+(p) \cup N_-(p))$, and assume that $H_{\operatorname{Re}p}(m)$, $H_{\operatorname{Im}p}(m)$ and the radial direction at $m$ are linearly independent. If $V$ is a sufficiently small neighborhood of $m$ in the two dimensional bicharacteristic strip through $m$ and $\Gamma$(resp. $\Gamma'$) is the cone generated by $\overline{V}$ (resp. $\partial V$), then one can for $v = 0, 1, \ldots$ find $u \in C^v(X)$ so that $WF(u) = \Gamma$, $WF(Pu) \subset \Gamma'$.*

It is easy to prove a global version of this result analogous to Theorem 3.2.3, at least when $V$ is simply connected. (For more precise results see Duistermaat - Hörmander [1]).

When the radial direction lies in the bicharacteristic two plane it seems hard to give simple general results. However, the following theorem contains a case discussed by Trèves [5, 7]. For the sake of simplicity we assume that the symbol of $P$ is an asymptotic sum of homogeneous terms.

THEOREM 3.4.3. *Let $\Lambda \subset p^{-1}(0)$ be a conic Lagrangean manifold and assume that on $\Lambda$ the projection of $H_p$ on the tangent space of $S^*(X)$ is proportional to a real vector and $\neq 0$. Let $\Gamma$ be the cone generated by a finite solution interval of this vector field which is not a closed curve in $S^*(X)$, and let $\Gamma'$ be generated by the end points of the interval. Then one can for $\nu = 0, 1, \ldots$ find $u \in C^\nu(Y)$ so that $WF(u) = \Gamma$ and $WF(Pu) \subset \Gamma'$.*

Note that $H_p$ is tangential to $\Lambda$ so the real vector field on $S^*(X)$ assumed to exist must be tangential to the submanifold of $S^*(X)$ induced by $\Lambda$. The proof of Theorem 3.4.3 is a repetition of that of Theorem 3.2.3 if one notes that for a homogeneous symbol differentiation in the radial direction is equivalent to multiplication by the degree. The first order differential equation in the direction $H_p$ occurring in the recursive determination of the amplitude can therefore be reduced to a differential equation with real coefficients.

Assuming the conjecture stated at the end of section 3.3, Trèves [7] deduced from the preceding results necessary conditions for hypoellipticity of *differential* operators $P$ with non-singular characteristics which were also proved to be sufficient. If $P$ is such an operator, the necessary conditions are derived as follows:

a) By Theorem 3.3.7 we must have $N_-(p) = \varnothing$, hence $N_+(p) = N_-(p)' = \varnothing$.

b) By Theorem 3.4.2 the projection in $T(X)$ of $H_p$ must have a real direction if $p = 0$. (If $P$ is strictly hypoelliptic we conclude that $H_p$ itself must have a real direction modulo the radial direction. In view of Theorem 3.4.3 we then obtain a contradiction if $H_p$ does not have a real direction at some point.) Assuming from now on that $p'_\xi \neq 0$ when $p = 0$ we obtain, if $H_{\text{Re}\,p}(m)$, $H_{\text{Im}\,p}(m)$ are linearly independent for some $m$ with $p(m) = 0$, that the projection $p^{-1}(0) \to X$ has rank $n - 1$ at every point in some neighborhood of $m$. The projection is therefore a hypersurface $Y$, defined by an equation $\rho(x) = 0$ with grad $\rho \neq 0$. Since $\rho$ vanishes on $p^{-1}(0)$ near $m$ it follows that $H_\rho$ is a linear combination of $H_{\text{Re}\,p}$ and $H_{\text{Im}\,p}$. Hence $p(m') = 0$ implies

$p(m' + tH_p(m')) = 0$ if $t$ is small and $m'$ is close to $m$. But $p$ is a polynomial in the fibers so this must be an identity in $t$. Thus $p$ must vanish in the normal bundle $N(Y)$ of $Y$, which is a Lagrangean manifold. On $N(Y)$ we also obtain that $H_p$ is a linear combination of $H_{\mathrm{Re}\,p}$ and $H_{\mathrm{Im}\,p}$ which means that the hypotheses of Theorem 3.4.3 are fulfilled so that $P$ cannot be hypoelliptic. This contradiction shows that indeed $H_p$ must be proportional to a real vector.

c) By Theorem 3.4.1 there cannot exist any one dimensional bicharacteristic strip for $p$. Hence it follows from b) that $\mathrm{Im}\,p$ cannot vanish on an interval of a bicharacteristic strip for $\mathrm{Re}\,p$.

d) Let $p(m) = 0$ and assume that $H_{\mathrm{Re}\,p}(m) \neq 0$. If the conjecture at the end of section 3.3 is true, it follows that on each bicharacteristic strip of $\mathrm{Re}\,p$ in a neighborhood of $m$ the restriction of $\mathrm{Im}\,p$ is everywhere $\leqq 0$ or everywhere $\geqq 0$. Only one of the cases can occur for otherwise there would exist a bicharacteristic strip for $\mathrm{Re}\,p$ on which $\mathrm{Im}\,p$ vanishes, in contradiction with c). Hence we conclude that either $\mathrm{Im}\,p \geqq 0$ in a neighborhood of $m$ when $\mathrm{Re}\,p = 0$, or else the opposite inequality is valid. Since we can choose $a \in C^\infty$ near $m$ so that $a\,\mathrm{Re}\,p + \mathrm{Im}\,p$ is constant on a vector field transversal to $(\mathrm{Re}\,p)^{-1}(0)$, this means that $m$ belongs to the set $N_U(p)$ introduced in

DEFINITION 3.4.4. We shall denote by $N_U(p)$ the set of all $m \in T^*(X)\backslash 0$ such that for some $C^\infty$ function $q$ in a neighborhood of $m$ we have $q(m) \neq 0$ and $\mathrm{Im}\,qp \geqq 0$.

Naturally the function $q$ can be chosen homogeneous. The set $N_U(p)$ is open and contains the complement of $p^{-1}(0)$. Only the intersection with $p^{-1}(0)$ is therefore interesting and it might have been more appropriate to introduce only this set in the definition. Note that $N_U(p) \cap N_+(p) = N_U(p) \cap N_-(p) = \varnothing$ for any $p$.

Modulo the truth of the conjecture at the end of section 3.3 it is therefore proved that if $p$ is hypoelliptic and $p'_\xi \neq 0$ when $p = 0$ then $N_U(p) = T^*(X)\backslash 0$ and there is no one dimensional bicharacteristic strip for $p$ (condition c) above). Conversely, Trèves [7] also proved that these conditions imply hypoellipticity. We shall give a proof in the following section where we also study the wave front set of solutions of $Pu = f$ in $N_U(p)$.

## 3.5. *Sufficient conditions for solvability and hypoellipticity*

Apart from the results of Hörmander [3] and Egorov [2] already referred to, all such conditions given so far in the literature include the assumption

$$N_+(p) \cup N_-(p) = \varnothing . \tag{3.5.1}$$

This is a necessary condition in the case of differential operators but not in general. (Cf. Definition 3.3.6 and Theorem 3.3.7.) When (3.5.1) is fulfilled, $p$ is *real analytic*, and $p = 0$ implies $p'_\xi \neq 0$, Nirenberg and Trèves [2] have proved that $P$ is solvable at every point. In fact, they showed that for every $x_0 \in X$ and $s \in \mathbf{R}$ there is an open neighborhood $V$ of $x_0$ such that for every $f \in H_{(s)}(X)$ one can find $u \in H_{(s+\mu-1)}(X)$ with $Pu = f$ in $V$. The analyticity assumption is needed to give control of the changes of signs in say $\operatorname{Im} p$ when $\operatorname{Re} p = 0$. Unfortunately the proof which is based on an abstract version of the energy integral method does not seem to lead to information concerning the propagation of singularities. For this reason we content ourselves here with a reference to part II of Nirenberg-Trèves [2] and subsequent additions to appear in the same journal.

However, in $N_U(p)$ the situation is not too different from the real case studied in section 3.2. In fact, Trèves [7] has succeeded in extending the geometrical optics constructions to operators with $N_U(p) = T^*(X)\backslash 0$. The main point is that, although there may be no strict solutions to the characteristic and transport equations, it is possible to find sufficiently good approximate solutions. From his proof one can also obtain information on the wave front sets. We shall indicate a different approach here based on the energy integral method which gives a shorter though less constructive proof.

PROPOSITION 3.5.1. *Let $u \in \mathscr{D}'(X)$ and $Pu = f$, and consider a bicharacteristic strip $I \ni t \to \gamma(t) \in T^*(X)\backslash 0$ for* $\operatorname{Re} p$ *where $I = \{ t \in \mathbf{R};\, t_1 \leqq t \leqq t_2 \}$. Assume that $\operatorname{Im} p \geqq 0$ in a neighborhood of $\gamma(I)$. If $\gamma(I) \cap WF(f) = \varnothing$ and $\gamma(t_2) \notin WF(u)$, it follows that $\gamma(I) \cap WF(u) = \varnothing$. More precisely, if $f \in H_{(s)}$ at $\gamma(I)$ and $u \in H_{(s+\mu-1)}$ at $\gamma(t_2)$, then $u \in H_{(s+\mu-1)}$ at $\gamma(I)$.*

*Proof.* The assertion about $WF(u)$ follows from the last statement applied not only to $\gamma(I)$ but also to bicharacteristic strips for $\operatorname{Re} p$ nearby. In proving the last statement we may assume that $u \in H_{(s+\mu-3/2)}$ at $\gamma(I)$. It is convenient to assume that $\mu = 1$ which can be brought about by multiplication of $P$ to the left by an elliptic operator of order $1 - \mu$. Choose a closed conic neighborhood $\Gamma$ of $\gamma(I)$ such that $\operatorname{Im} p \geqq 0$ in a neighborhood of $\Gamma$, $f \in H_{(s)}$ and $u \in H_{(s-1/2)}$ in $\Gamma$. It is clearly enough to prove Proposition 3.5.1 locally so we may assume that $\Gamma$ has a compact projection in a coordinate patch which is identified with $\mathbf{R}^n$ and that $u \in \mathscr{E}'(\mathbf{R}^n)$.

Let $M \subset S^{s-1}(X \times \mathbf{R}^n)$ be a bounded subset of $S^s(X \times \mathbf{R}^n)$ which consists only of real valued functions with support in $\Gamma$. (We shall make an explicit choice of $M$ later where the closure in $S^s$ (in a weak topology) can contain symbols of order $s$.) With $c \in M$ we put $C = c(x, D)$ and form

$$(3.5.2) \qquad (Cf, Cu) = (CPu, Cu) = (PCu, Cu) + ([C, P]u, Cu).$$

Here ( , ) denotes the usual sesquilinear scalar product. Write $P = A + iB$ with $A$ and $B$ self-adjoint, that is, $A = (P+P^*)/2$, $B = (P-P^*)/2i$. The principal symbols $a$ and $b$ of $A$ and $B$ are $\operatorname{Re} p$ and $\operatorname{Im} p$ respectively. Taking the imaginary part of (3.5.2) we obtain

$$(3.5.3) \qquad \operatorname{Im}(Cf, Cu) = (BCu, Cu) + \operatorname{Re}([C, B]u, Cu) + \\ + \operatorname{Im}([C, A]u, Cu).$$

We can write $B = B_0 + B_1$ where the principal symbol of $B_0$ is non-negative everywhere and $WF(B_1)$ does not meet $\Gamma$. By a well known improvement of Gårding's inequality (Hörmander [3, Theorem 1.3.3]; see also Lax-Nirenberg [1], Kumano-go [1], Vaillancourt [1], and for a still more precise result Melin [1]) we have

$$(3.5.4) \qquad \operatorname{Re}(B_0 v, v) \geqq -C_1 \|v\|_{(0)}^2, v \in C_0^\infty,$$

where $\| \ \|_{(0)}$ is the norm in $L^2 = H_{(0)}$. (We use here the more restrictive definition of $H_{(s)}(\mathbf{R}^n)$ as $(1-\Delta)^{-s/2} L^2(\mathbf{R}^n)$.) Since $B_1 C$ is of order $-\infty$ we obtain with a constant $C_2$ depending on $u$ but not on $C$

$$(3.5.5) \qquad (BCu, Cu) \geqq -C_1 \|Cu\|_{(0)}^2 - C_2.$$

Next we note that the symbol of $C^*[C, B]$ is $ic\{b, c\} = i\{b, c^2\}/2$ apart from an error which belongs to a bounded set in $S^{2s-1}$. Since $\{b, c^2\}$ is real valued it follows that the symbol of the sum of $C^*[C, B]$ and its adjoint is in a bounded set in $S^{2s-1}$, which shows that with another $C_2$ depending on $u$

$$(3.5.6) \qquad \operatorname{Re}([C, B]u, Cu) \geqq -C_2.$$

In the same way we obtain

(3.5.7) $2\,\mathrm{Im}\big([C, A]\,u, Cu\big) \geqq \mathrm{Re}\big(\{a, c^2\}(x, D)\,u, u\big) - C_2\,.$

Summing up (3.5.3)-(3.5.7) we obtain with still another $C_2$

(3.5.8) $\mathrm{Re}\big(e(x, D)\,u, u\big) \leqq \|\,Cf\,\|_{(0)}{}^2 + C_2, C \in M\,,$

where

(3.5.9) $e(x, \xi) = \{a, c^2\}(x, \xi) - (2C_1 + 1)\,c(x, \xi)^2\,.$

Clearly $\|\,Cf\,\|_{(0)}$ is bounded when $C \in M$. Note that while $C_2$ and this bound may depend on $M$, the constant $C_1$ comes from (3.5.5) and is completely independent of the choice of $M$.

We may assume that the map from $I$ to the cosphere bundle defined by $\gamma$ is injective. Let $\Gamma_0$ be an open conic neighborhood of $\gamma(t_2)$ where $u \in H_{(s)}$ and choose a non-negative $C^\infty$ homogeneous function $c$ of degree $s$ with support in $\Gamma$ such that $\{a, c^2\} = H_{\mathrm{Re}\,p}\,c^2 \geqq 0$ in $\Gamma \backslash \Gamma_0$ with strict inequality in $\gamma(I) \backslash \Gamma_0$. That this is possible is seen immediately if we first define $c(x, \xi)$ for $|\,\xi\,| = 1$ using a norm in $T^*(X)$ which is constant on the integral curves of $H_{\mathrm{Re}\,p}$. Also choose $C^\infty$ functions $a_0$ and $a_1$ homogeneous of degree 0 and 1 respectively so that $H_a a_0 = 1$, $H_a a_1 = 0$ and $a_1$ is different from 0 in the support of $c$. This is also possible if the support of $c$ is a sufficiently small neighborhood of $\gamma(I)$. Now $M$ will consist of the functions

$$c_{\lambda,\varepsilon} = c\,e^{\lambda a_0}(1 + \varepsilon^2 a_1{}^2)^{-1/2}, 0 < \varepsilon \leqq 1\,,$$

where $\lambda$ is *fixed* $\geqq C_1 + 1$. If $c$ is replaced by $c_{\lambda,\varepsilon}$ the function $e$ in (3.5.9) becomes

$$e_{\lambda,\varepsilon} = \big(\{a, c^2\} + (2\lambda - 2C_1 - 1)\,c^2\big)\,e^{2\lambda a_0}(1 + \varepsilon^2 a_1{}^2)^{-1}\,.$$

Since $e_{\lambda,\varepsilon} \geqq 0$ outside $\Gamma_0$ with strict inequality on $\gamma(I) \backslash \Gamma_0$ we can choose a non-negative homogeneous function $r$ of degree $s$ which is positive on $\gamma(I)$, and a real valued homogeneous function $q$ of order $s$ with support in $\Gamma_0$, thus $q(x, D)\,u \in L^2$, such that

(3.5.10) $r^2 \leqq \big(\{a, c^2\} + (2\lambda - 2C_1 - 1)\,c^2\big)\,e^{2\lambda a_0} + q^2\,.$

Let $r_\varepsilon = r\,(1 + \varepsilon^2 a_1^2)^{-1/2}$ and $q_\varepsilon = q\,(1 + \varepsilon^2 a_1{}^2)^{-1/2}$. An application of (3.5.4) to the operator with principal symbol equal to the difference of the two sides in (3.5.10) multiplied by $|\,\xi\,|^{1-2s}$ leads to the estimate

$$\|\,r_\varepsilon(x, D)\,u\,\|_{(0)}{}^2 \leqq \mathrm{Re}\big(e_{\lambda,\varepsilon}(x, D)\,u, u\big) + \|\,q_\varepsilon(x, D)\,u\,\|_{(0)}{}^2 + C_3$$

since $u \in H_{(s-1/2)}$ in $\Gamma$. (Here we rely on the uniformity of (3.5.4) when the symbol of $B_0$ is bounded in $S^1$.) In view of (3.5.8) we conclude that

$\| r_\varepsilon(x, D) u \|_{(0)}$ is bounded when $\varepsilon \to 0$, which proves that the limit $r(x, D) u$ of $r_\varepsilon(x, D) u$ in $\mathscr{D}'$ must belong to $L^2$. Hence $u \in H_{(s)}$ at $\gamma(I)$, which proves the proposition.

Another way of stating the proposition is that if $\gamma(I) \cap WF(f) = \varnothing$ and $\gamma(t_1) \in WF(u)$, then $\gamma(I) \subset WF(u)$. In view of Theorem 2.2.2 it follows that $\gamma(I) \subset p^{-1}(0)$, which implies that $H_{\operatorname{Im} p} = 0$ on $\gamma(I)$ since $\operatorname{Im} p \geqq 0$. Thus $\gamma(I)$ is a bicharacteristic strip for $p$. This gives the following extension of a result of Trèves [7] mentioned above:

THEOREM 3.5.2. *If $\Gamma$ is an open cone $\subset N_U(p)$ containing no bicharacteristic strip for $p$, then*

$$WF(Pu) \cap \Gamma = WF(u) \cap \Gamma, u \in \mathscr{D}'(X). \tag{3.5.10}$$

*If $\Gamma \supset p^{-1}(0)$ it follows that $P$ is strictly hypoelliptic.*

We can also obtain conclusions concerning the global existence of solutions and the global regularity question (3.4.2). To state them we first have to discuss the orientation of the Hamilton field $H_p(m)$ when $m \in N_U(p) \cap p^{-1}(0)$. According to Definition 3.4.4 we can choose $q$ so that $q(m) \neq 0$ and $\operatorname{Im} qp \geqq 0$ near $m$. With $p_1 = qp$ we have then $d \operatorname{Re} p_1(m) \neq$ $\neq 0$, $d \operatorname{Im} p_1(m) = 0$. If for another function $r$ with $r(m) \neq 0$ we have $\operatorname{Im} rp_1 \geqq 0$ near $m$, then $r(m)$ is either positive or negative. In the latter case we obtain $\operatorname{Im} p_1 \leqq 0$ near $m$ when $\operatorname{Re} p_1 = 0$, and since $\operatorname{Im} p_1 \geqq 0$ it follows that $\operatorname{Im} p_1 = 0$ near $m$ when $\operatorname{Re} p_1 = 0$. Hence $\operatorname{Im} p_1 = s \operatorname{Re} p_1$ for some smooth $s$, which means that $p_1 = (1+is) \operatorname{Re} p_1$ is real apart from a non-vanishing factor. If $p$ is not of this special form we conclude that $r(m) > 0$, hence that $H_{rp_1}(m) = r(m) H_{p_1}(m)$ has the same direction as $H_{p_1}(m)$.

DEFINITION 3.5.3. By $N_R(p)$ we denote the set of all $m \in T^*(X) \backslash 0$ such that there is a $C^\infty$ function $q$ in a neighborhood of $m$ with $q(m) \neq 0$ and $qp$ real.

$N_R(p)$ is of course an open subset of $N_U(p)$ containing the complement of $p^{-1}(0)$. In $N_R(p) \cap p^{-1}(0)$ there is no natural way of choosing a complex number $z$ such that $zH_p$ is real, but if $m \in N_U(p) \backslash N_R(p)$ we choose as positive the direction of $q(m) H_p(m)$ when $q(m) \neq 0$ and $\operatorname{Im} qp \geqq 0$ in a neighborhood of $m$. The arguments preceding Definition 3.5.3 proved precisely that this definition is unique.

In $N_R(p)$ we have the situation studied in section 3.2. However, the orientation of the Hamiltonian field in $N_U(p)\backslash N_R(p)$ enters the analogue of Theorem 3.2.1 there.

THEOREM 3.5.4. *Let $u \in \mathscr{D}'(X)$ and $Pu = f$. If $m \in (WF(u)\backslash WF(f)) \cap N_U(p)$, then there exists a bicharacteristic strip $I \ni t \to \gamma(t) \in N_U(p)\backslash WF(f)$ for $p$ with $m \in \gamma(I) \subset WF(u)$ such that $I$ is a (finite) interval on* **R** *and, if $t_0$ is a boundary point of $I$,*

(i) *$\gamma(t_0) \in N_U(p)\backslash N_R(p)$ and the positive direction of $H_p(\gamma(t_0))$ points towards $\gamma(I)$ if $t_0 \in I$.*

(ii) *$\gamma(t)$ does not converge to a limit in $N_U(p)\backslash WF(f)$ as $I \ni t \to t_0$ if $t_0 \notin I$.*

The proof follows from Proposition 3.5.1.

We can now give a partial extension of Theorem 3.2.4. Assume that $N_U(p) = T^*(X)\backslash 0$. We shall say that a curve $I \ni t \to \gamma(t) \in p^{-1}(0)$ is a complete bicharacteristic strip for $p$ if $I$ is a finite interval in **R** and

(i) $d\gamma/dt$ is proportional to $H_p(\gamma(t))$, $t \in I$,

(ii) $\gamma(t_0) \in N_U(p)\backslash N_R(p)$ and the positive direction of $H_p(\gamma(t_0))$ points towards $\gamma(I)$ if $t_0$ is a boundary point of $I$ belonging to $I$.

(iii) $\gamma(t)$ does not converge to a limit in $N_U(p)$ as $I \ni t \to t_0$ if $t_0 \notin I$.

THEOREM 3.5.5. *Assume that $N_U(p) = T^*(X)\backslash 0$ and that no complete bicharacteristic strip for $\bar{p}$ stays over a compact set in $X$. Every $u \in \mathscr{E}'(X)$ with $P^*u \in C^\infty(X)$ is then in $C_0^\infty(X)$, which implies that the equation $Pu = f$ can be solved in a neighborhood of any compact set $K \subset X$ when $f$ is orthogonal to the finite dimensional vector space of functions $v \in C_0^\infty(K)$ with $P^*v = 0$. The map from $\mathscr{D}'(X)$ to $\mathscr{D}'(X)\backslash C^\infty(X)$ defined by $P$ is surjective if in addition for every compact set $K \subset X$ there is another compact set $K' \subset X$ such that $K'$ contains the projection of any compact interval $I$ on a complete bicharacteristic strip $J$ for $\bar{p}$ with the projection of the boundary of $I$ relative to $J$ contained in $K$.*

The proof is a repetition of part of the proof of Theorem 3.2.4 with Theorem 3.2.1 replaced by Theorem 3.5.4.

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