Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 16 (1970)

Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON THE CLOSED CONVEX HULL OF A COMPACT SET

Autor: Goodner, Dwight B.

DOI: https://doi.org/10.5169/seals-43862

ON THE CLOSED CONVEX HULL OF A COMPACT SET

by Dwight B. GOODNER

Although it is known that the closed convex hull of a compact set may not be compact, the literature contains few, if any, reasonably easy proofs of this theorem, and existing proofs seem to be difficult to find. Unfortunately many students encounter this result before their training and experience have prepared them to prove it. The purpose of this note is to give a proof that seems to be minimal in terms of the knowledge and mathematical sophistication required to understand it.

Theorem. There is a linear topological space in which the closed convex hull of a certain compact set is not compact.

Proof. Let m be the space of bounded real-valued sequences with the usual supremum norm. We proceed by identifying a compact set S [2, p. 135] contained in a linear subspace L [1, p. 12] of m such that in L the convex hull [1, p. 10] of S is closed but is not compact.

Let ω be the directed system of positive integers. For fixed $i \in \omega$ let $\delta^i = \{ \delta^i{}_j \}_{j \in \omega}$ be the sequence defined by $\delta^i{}_i = 1$ and $\delta^i{}_j = 0$ for $i \neq j$; $\delta^i{}_j$ is the Kronecker delta. We note that $\delta^i \in m$. Let $S = \{ \Theta \} \cup \{ \frac{\delta^i}{i} : i \in \omega \}$ where Θ is the zero element in m, and let $K(S)$ be the convex hull of S in m. Since $\lim_{i \to \infty} \frac{\delta^i}{i} = \Theta$, each open neighborhood of Θ contains all but finitely many elements of the sequence $\{ \frac{\delta^i}{i} \}_{i \in \omega}$, and we see that S is compact.

Let

$$x_1 = 1 \frac{\delta^1}{1}$$

$$x_2 = \tfrac{1}{2} \frac{\delta^1}{1} + \tfrac{1}{2} \frac{\delta^2}{2}$$

$$x_3 = \tfrac{1}{2}\frac{\delta^1}{1} + \tfrac{1}{4}\frac{\delta^2}{2} + \tfrac{1}{4}\frac{\delta^3}{3}$$

. . .

$$x_i = \tfrac{1}{2}\frac{\delta^1}{1} + \tfrac{1}{4}\frac{\delta^2}{2} + \cdots + \frac{1}{2^{i-1}}\frac{\delta^{i-1}}{i-1} + \frac{1}{2^{i-1}}\frac{\delta^i}{i}$$

. . .

For each i, $x_i \in K(S)$ [1, p. 10]. If

$$x_0 = \lim_{i\to\infty} x_i = \lim_{i\to\infty} \Big(\sum_{n=1}^{i-1} \frac{1}{2^n}\frac{\delta^n}{n} + \frac{1}{2^{i-1}}\frac{\delta^i}{i}\Big) = \lim_{i\to\infty} \sum_{n=1}^{i} \frac{1}{2^n}\frac{\delta^n}{n},$$

then $x_0 \notin K(S)$ but $x_0 \in \mathrm{cl}\, K(S)$ where $\mathrm{cl}\, K(S)$ is the closure of $K(S)$ in m. We remark that this shows that in a linear topological space the convex hull of a convex set may not be closed and, hence, may not be compact [2, p. 141]. We note but do not prove that the closed convex hull, $\mathrm{cl}\, K(S)$, of S is compact.

Let L be the set of all linear combinations of all finite subsets of S. Then L with the relative topology of m [2, p. 51] is the smallest linear subspace [1, p. 2] of m containing $K(S)$, and $K'(S) = L \cap \mathrm{cl}\, K(S)$ is the closed convex hull of S in L. To complete our proof it suffices to show that $K'(S)$ is not a compact subset of L.

The sequence $\{x_i\}_{i\in\omega}$ defined above is a closed subset of L since its only cluster point $x_0 \notin L$. For each $k \in \omega$ let $A_k = \{x \in L : x \notin \{x_i\}_{i=k}^{\infty}\}$. Each A_k is an open subset of L and $A = \{A_k : k \in \omega\}$ is an open cover of $K'(S)$. However, if $A_{k(1)}, A_{k(2)}, \ldots, A_{k(n)}$ is a finite subset of A and if j is any integer greater than $\max\{k(1), k(2), \ldots, k(n)\}$, then $x_j \notin U_{i=1}^{n} A_{k(i)}$. Since $x_j \in \{x_i\}_{i\in\omega} \subset K'(S)$ it follows that A does not contain a finite subset which covers $K'(S)$ and, hence, that $K'(S)$ is not a compact subset of L which completes the proof.

REFERENCES

[1] M. M. Day, *Normed linear spaces.* Academic Press, New York and Springer, Berlin, 1962.

[2] J. L. Kelley, *General topology.* D. Van Nostrand Co., Princeton, 1955.

D. B. GOODNER
Dept. of Math.
Florida State University
Tallahassee, Florida 32306