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A NEW EXTENSION OF HÖLDER'S INEQUALITY

by J. A. CARROLL, R. CORDNER and C. J. A. EVELYN

1. We prove

$$[\Sigma(AB \dots L)^{ab \dots l}]^{1/ab \dots l} \leq [\Sigma A^{\alpha a^\alpha}]^{1/\alpha a^\alpha} \cdot [\Sigma B^{\beta b^\beta}]^{1/\beta b^\beta} \dots [\Sigma L^{\lambda l^\lambda}]^{1/\lambda l^\lambda} \quad (1)$$

Where $A, B, \dots L$ are sets of non-negative real numbers, all sets having the same number of members and $a, \alpha, b, \beta, \dots l, \lambda$ are positive real numbers with

$$1/\alpha + 1/\beta + \dots + 1/\lambda = 1/k.$$

The inequality holds, writing p for $(ab \dots l)$

- (a) When $k = 1$ for all p
- (b) When $k < 1$ if $p \geq k^{1/(1-k)}$
When $k > 1$ if $p \leq k^{1/(1-k)}$

There is equality if and only if

- (a) every number in one of the sets $A, B, \dots L$ or all but one is zero, and in the latter case those which are positive have the same rank.

Or

- (b) $a^\alpha = b^\beta = \dots = l^\lambda$ and the sets $A^\alpha, B^\beta, \dots L^\lambda$ are proportional, and $k = 1$ or, if $k \neq 1$, $p = k^{1/(1-k)}$.

A variant of the result (1) is given in paragraph 7.

- 2) A well known extension of Hölder's Inequality [1], [5] is, in our notation [2]

$$\Sigma[AB \dots L] \leq [\Sigma A^\alpha]^{1/\alpha} \cdot [\Sigma B^\beta]^{1/\beta} \dots [\Sigma L^\lambda]^{1/\lambda} \quad (2)$$

where $\Sigma 1/\alpha = 1$, which is (1) for $k = 1$, $p = 1$.

Less commonly cited is a further extension by Jensen [3], [6] of 2 for $k < 1$, $p = 1$.

A rather different type of extension has recently been given by Daykin and Eliezer [7].

Our result extends Jensen's by covering the cases $k = 1$, all p and determining the restrictions on p when $k \neq 1$.

4) To prove our result we first transform the inequality (1) by writing p for $(ab \dots l)$ to obtain

$$\Sigma[AB \dots L] \leq [\Sigma A^{\alpha a^\alpha}]^{p/\alpha a^\alpha} \dots [\Sigma L^{\lambda l^\lambda}]^{p/\lambda l^\lambda}$$

which holds when

$$\Sigma p/\alpha a^\alpha \geq 1 \quad (3)$$

by Jensen's Theorem [3] and Hölder's Inequality [1].

Now if

$$\Sigma 1/\alpha = 1/k \quad (4)$$

$$\begin{aligned} \Sigma p/\alpha a^\alpha &= (p/k) \Sigma k/\alpha a^\alpha \\ &\geq (p/k) \cdot 1/(a^k b^k \dots l^l) \\ &= p/kp^k \end{aligned}$$

by Arithmetic Mean \geq Geometric Mean [4].

$$\left. \begin{aligned} \text{Thus } 1 \text{ holds when } p/kp^k \geq 1. \\ \text{This is so when } k < 1 \text{ if } p \geq k^{1/(1-k)} \\ \text{and when } k > 1 \text{ if } p \leq k^{1/(1-k)} \end{aligned} \right\} \quad (6)$$

The conditions for equality are readily seen.

- 5) If the conditions (6) are not met, the inequality (1) may still hold, as it is possible for it to be true when $\Sigma p/\alpha a^\alpha$ is not ≥ 1 , and there is no simple test to cover this case: our conditions, when met, assure that $\Sigma p/\alpha a^\alpha \geq 1$.
 6) Some special cases of interest are:

If $p = 1$ 1 holds for any $k \leq 1$.

$p < 1/e$ 1 holds if $k > 1$.

$p > 1/e$ 1 holds if $k < 1$.

7. A variant of (1) is

$$\begin{aligned} &[(\mathcal{A}\mathcal{B}\dots\mathcal{L})^p - \Sigma(AB \dots L)^p]^{1/p} \\ &\geq [\mathcal{A}^{\alpha a^\alpha} - \Sigma A^{\alpha a^\alpha}]^{1/\alpha a^\alpha} \dots [\mathcal{L}^{\lambda l^\lambda} - \Sigma L^{\lambda l^\lambda}]^{1/\lambda l^\lambda} \end{aligned} \quad (7)$$

To obtain this we consider

$$\Sigma(AB \dots L)^p + (\mathcal{A}^{\alpha a^\alpha} - \Sigma A^{\alpha a^\alpha})^{p/\alpha a^\alpha} + +$$

which by (1) is

$$\leqq (\Sigma A^{\alpha a^\alpha} + \mathcal{A}^{\alpha a^\alpha} - \Sigma A^{\alpha a^\alpha})^{p/\alpha a^\alpha} \dots$$

which gives (7).

8. Analogous integral inequalities only exist [8] when $\Sigma 1/\alpha a^\alpha = 1/p$, and then (1) reduces to one form of Holder's Inequality for which the integral analogue is well known.

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