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### 3. APPLICATIONS

In this section we want to indicate how Theorems 1.1–1.5 come out as corollaries of Theorem 2.1. Some other consequences of this theorem will also be mentioned. Furthermore, we give references concerning equations (1.1)–(1.5) and we review some geometric applications of the relevant uniqueness theorems.

In order to get the equations (1.1)–(1.3) from (1.6) one has to choose the measure  $\varphi$  appropriately. The condition, e.g., that (1.1) be true for any  $d$  pairwise orthogonal unit vectors  $u_1, \dots, u_d \in \Omega_d$  is equivalent to the equation

$$(3.1) \quad f(\delta u_1) + \dots + f(\delta u_d) = 0 \quad \text{for each } \delta \in SO(d),$$

where now  $u_1, \dots, u_d$  is a fixed  $d$ -tuple of pairwise orthogonal unit vectors. Equation (3.1) results from (1.6) if  $\varphi$  is the discrete measure concentrated in  $u_1, \dots, u_d$  and assigning the same weight to each of these points. If now (3.1) holds, then Theorem 2.1 shows that  $(f, Y_{ni}) = 0$  ( $i = 1, \dots, N_{d,n}$ ) must hold for each  $n \in \{0, 1, 2, \dots\}$  for which

$$Y_n(u_1) + \dots + Y_n(u_d) \neq 0$$

for some spherical harmonic  $Y_n$  of degree  $n$ . If we choose for  $Y_n$  the zonal harmonic defined by  $Y_n(u) = C_n^v(\langle u_1, u \rangle)$ , where  $C_n^v$  is the Gegenbauer polynomial of degree  $n$  and order  $v = \frac{1}{2}(d-2)$ , then we get

$$\sum_{r=1}^d Y_n(u_r) = C_n^v(1) + (d-1) C_n^v(0) \neq 0 \quad \text{for } n \neq 2.$$

From the completeness of the system of spherical harmonics we conclude  $f \in \mathfrak{H}_2$ . That each element of  $\mathfrak{H}_2$  is in fact a solution of (3.1) may be shown directly.

Theorem 1.1 is, for  $d = 3$ , due to Blaschke [3], who used it to derive the following geometric result: *If the vertices of the boxes circumscribed to a given convex body  $K \subset E^3$  lie on some fixed sphere, then  $K$  is a solid ellipsoid.* Chakerian [7] has generalized this theorem by induction with respect to the dimension and has drawn another geometric consequence.

Theorem 1.2 is proved similarly, though in this case the decision whether for given  $n$  the inequality

$$Y_n(u_1) + \dots + Y_n(u_{d+1}) \neq 0$$

is valid for some  $Y_n \in \mathfrak{H}_n$  or not, is a bit more difficult. For  $d = 3$ , Theorem 1.2 is due to Meissner [13]; for general  $d$  it is contained in [19] (Hilfsatz 4.4). The geometric problem leading to equation (1.2) and more generally to equations of type (1.6) with discrete  $\varphi$ , can be described as follows: Let  $P \subset E^d$  be a  $d$ -dimensional convex polytope; a convex body  $K \subset E^d$  is called a *rotor* of  $P$  if to each  $\delta \in SO(d)$  there exists a translation  $\tau$  of  $E^d$  such that  $\tau\delta K$  is contained in the polytope  $P$  and touches each of its  $(d-1)$ -dimensional faces. Roughly speaking, a rotor can be completely turned inside  $P$ , always gliding along its facets. As an example, we mention the bodies of constant width one, which are the rotors of the unit cube. Meissner [13] has determined all the nontrivial (i.e. non-spherical) rotors of the threedimensional regular polyhedra. The general problem of determining all pairs  $(P, K)$  where  $P \subset E^d$  is a  $d$ -dimensional convex polytope (not necessarily bounded) and  $K$  is a nontrivial rotor of  $P$ , has been completely solved in [19].

In order to obtain Theorem 1.3 we choose  $\varphi$  as the measure concentrated on a fixed great sphere  $s_a$  and proportional to the  $(d-2)$ -dimensional Lebesgue measure on  $s_a$ . Then (1.6) holds if and only if (1.3) holds for each  $v \in \Omega_d$ . For a spherical harmonic  $Y_n \in \mathfrak{H}_n$  an easy computation gives (see [17], formula (5); put  $\alpha = 0$ )

$$(Y_n, \varphi) = \int_{s_a} Y_n d\lambda_a = \omega_{d-1} C_n^v(1)^{-1} C_n^v(0) Y_n(a),$$

where  $\omega_{d-1}$  is the surface area of the unit sphere in  $E^{d-1}$  (this equation can also be derived by a limit process from the Funk-Hecke-formula; compare (3.4) below). If  $Y_n$  is properly chosen, this is  $\neq 0$  for even  $n$ . Thus by Theorem 2.1, a function satisfying

$$\int_{s_a} f(\delta u) d\lambda_a = 0 \quad \text{for each } \delta \in SO(d)$$

must be orthogonal to each spherical harmonic of even degree, hence if  $f$  itself is even, it must vanish identically.

For  $d = 3$ , Theorem 1.3 is due to Minkowski [14], who used it to prove that *a convex body of constant girth is also a body of constant width*. Other proofs of Theorem 1.3 for  $d = 3$  may be found in Funk [8], Bonnesen-Fenchel [6] (p. 136-138); proofs for  $d \geq 3$  have been given by Petty [16] and Schneider [17]. Funk [8] (p. 287) remarked the following geometric consequence of Theorem 1.3: *The spherical ball is the only centrally symmetric convex body with the property that all intersections of the body with*

planes through its center have the same surface area. Other geometric applications of Theorem 1.3 may be found in [2], [4], [12], [20].

In order to get the equations (1.4) and (1.5) from (1.6) one has to choose the function  $f$  appropriately. Both equations can be written in the form

$$(3.2) \quad \int_{\Omega_d} g(\langle u, v \rangle) d\varphi(u) = 0 \quad \text{for each } v \in \Omega_d,$$

where  $g(t) = |t|$  in the case of (1.4), and  $g(t) = 1$  for  $t > 0$  and  $= 0$  for  $t \leq 0$  in the case of (1.5). If we put  $f(u) = g(\langle u, a \rangle)$  for some fixed  $a \in \Omega_d$ , then (3.2) is equivalent to

$$\int_{\Omega_d} f(\delta u) d\varphi(u) = 0 \quad \text{for each } \delta \in SO(d).$$

If now (3.2) holds, then Theorem 2.1 shows that  $(Y_{ni}, \varphi) = 0$  ( $i = 1, \dots, N_{d,n}$ ) must hold for each  $n \in \{0, 1, 2, \dots\}$  for which

$$(3.3) \quad \int_{\Omega_d} g(\langle u, a \rangle) Y_n(u) d\omega(u) \neq 0$$

for some spherical harmonic  $Y_n$  of degree  $n$ . Here we should observe that  $g$  is only piecewise continuous in the case of equation (1.5), so that we cannot apply Theorem 2.1 verbally. It is, however, not difficult to generalize Theorem 2.1 appropriately. In order to decide whether (3.3) holds we apply the Funk-Hecke-formula (see, e.g., Müller [15], p. 20)

$$(3.4) \quad \int_{\Omega_d} g(\langle u, a \rangle) Y_n(u) d\omega(u) \\ = \omega_{d-1} C_n^v(1)^{-1} \int_{-1}^1 g(t) C_n^v(t) (1-t^2)^{\frac{1}{2}(d-3)} dt Y_n(a).$$

If we now make the additional assumption that  $(Y_{ni}, \varphi) = 0$  ( $i = 1, \dots, N_{d,n}$ ) for each  $n$  for which

$$(3.5) \quad \int_{-1}^1 g(t) C_n^v(t) (1-t^2)^{\frac{1}{2}(d-3)} dt = 0,$$

then we get  $(Y_{ni}, \varphi) = 0$  ( $i = 1, \dots, N_{d,n}$ ) for  $n = 0, 1, 2, \dots$  and hence  $(h, \varphi) = 0$  for each continuous function  $h$  on  $\Omega_d$ , which shows that  $\varphi = 0$ . In special applications this additional assumption turns out to be less formal than it might seem. For instance, if  $g(t) = |t|$ , then (3.5) holds exactly for odd  $n$ , hence it suffices to assume that  $\varphi$  is even. Thus Theorem 1.4 comes out. Theorem 1.5 is obtained similarly.

Theorem 1.4, for  $d = 3$  and special measures, may be found in the book of Blaschke [5] (p. 152, 154-155). The general case is due to A. D. Aleksandrov [1] (§8) (though in the form of a seemingly more special geometric theorem); compare also Petty [16] (p. 1545-1546). The case  $d = 3$  (and  $\varphi$  specialized) of Theorem 1.5 is due to Funk [9]; another proof (of special cases in geometric formulation) has been given by Kubota [11]. The common generalization of both uniqueness theorems, which is given above, may be found in [18]. To this paper we refer also for references to the known geometric consequences of Theorems 1.4 and 1.5, as well as for some new applications thereof.

The question leading to Theorem 1.5 can be generalized in the following way: Let  $D \subset \Omega_d$  be any domain, and let us say for the moment that  $D$  is *non-special* if and only if every measure  $\varphi$  on  $\Omega_d$  for which  $\varphi(D') = 0$  for each domain  $D'$  (properly) congruent to  $D$ , must vanish identically; otherwise  $D$  is called *special*. If  $D$  is a spherical cap of radius  $\alpha \in (0, \pi)$ , it has been shown that  $D$  is special if and only if  $\alpha$  is contained in a certain set of values which is denumerable and dense in  $(0, \pi)$  (Ungar [21], more general in [18]). Ungar [21] has given an example of a non-circular special domain on  $\Omega_3$ . Now Theorem 2.1 (if generalized to piecewise continuous functions) allows, at least theoretically, to decide whether a given domain  $D \subset \Omega_d$  is non-special: For this to be the case it is necessary and sufficient that

$$\int_D Y_{ni} d\omega \neq 0$$

for each  $n \in \{0, 1, 2, \dots\}$  and some  $i \in \{1, \dots, N_{d,n}\}$ . Thus the answer depends on the computation of denumerably many definite integrals.

Finally we mention that a special case of the 2-dimensional analogue of Theorem 2.1 was used by Görtler [10] in characterizing those pairs of plane convex domains whose mixed area is invariant under arbitrary (non-simultaneous) motions of the domains.

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