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Autor: Schneider, Rolf

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FUNCTIONAL EQUATIONS CONNECTED WITH ROTATIONS AND THEIR GEOMETRIC APPLICATIONS

by Rolf SCHNEIDER

1. INTRODUCTION

In the geometry of convex bodies there is a certain type of questions which lead to some linear functional equations on the Euclidean sphere. Among these equations are, e.g., certain integral equations as well as relations between the values of a function at certain sets of finitely many arguments. What we mean by " linear functional equations on the sphere " is perhaps best described by giving some typical results:

Let E^d $(d \geqq 3)$ denote d-dimensional Euclidean space with scalar product $\langle\ ,\ \rangle$, and let

$$\Omega_d := \{ x \in E^d \mid \langle x, x \rangle = 1 \}$$

be its unit sphere.

THEOREM 1.1. *Let* f *be a real continuous function on* Ω_d *satisfying*

$$f(u_1) + \ldots + f(u_d) = 0 \tag{1.1}$$

for any d *pairwise orthogonal vectors* $u_1, \ldots, u_d \in \Omega_d$. *Then* f *is a spherical harmonic of degree* 2.

THEOREM 1.2. *Let* f *be a real continuous function on* Ω_d *satisfying*

$$f(u_1) + \ldots + f(u_{d+1}) = 0 \tag{1.2}$$

whenever $u_1, \ldots, u_{d+1}$ *are the vertices of a regular simplex inscribed in* Ω_d. *Then* f *is a sum of spherical harmonics of degrees* 0, 1, 2, 5 *if* $d = 3$, *and of degrees* 0, 1, 2 *if* $d \geqq 4$.

For $v \in \Omega_d$ let

$$s_v := \{ u \in \Omega_d \mid \langle u, v \rangle = 0 \}$$

be the great sphere with pole v; and let λ_v denote the $(d-2)$-dimensional Lebesgue measure on s_v.

THEOREM 1.3. *If* f *is a real, even, continuous function on* Ω_d *satisfying*

$$\int_{S_v} f \, d\lambda_v = 0 \tag{1.3}$$

for each $v \in \Omega_d$, *then* f $= 0$.

In the following, by a measure on Ω_d we understand a real valued, countably additive set function, defined on the σ-field of Borel subsets of Ω_d. A measure φ on Ω_d is called *even* (respectively *odd*) if $\varphi(B) = \varphi(B^*)$ $(\varphi(B) = -\varphi(B^*))$ for any two antipodal Borel subsets B, B^* of Ω_d.

THEOREM 1.4. *If* φ *is an even measure on* Ω_d *satisfying*

$$\int_{\Omega_d} |\langle u, v\rangle| \, d\varphi(u) = 0 \tag{1.4}$$

for each $v \in \Omega_d$, *then* $\varphi = 0$.

For $v \in \Omega_d$ let

$$S_v := \{ u \in \Omega_d \mid \langle u, v \rangle > 0 \}$$

be the open hemisphere with center v.

THEOREM 1.5. *If* φ *is an odd measure on* Ω_d *satisfying*

$$\varphi(S_v) = 0 \tag{1.5}$$

for each $v \in \Omega_d$, *then* $\varphi = 0$.

To functional relations of the types (1.1)—(1.5) one is lead by some uniqueness and characterization problems in the theory of convex bodies; and the theorems quoted above (all of which are essentially known—see section 3) have interesting geometric interpretations. The main purpose of this note is to treat the above functional equations from a unifying point of view and to exhibit them as special cases of one general equation.

In fact, each of these equations can be written in the form (see section 3)

$$\int_{\Omega_d} f(\delta u) \, d\varphi(u) = 0 \qquad \text{for each } \delta \in SO(d). \tag{1.6}$$

Here $SO(d)$ denotes the d-dimensional rotation group acting on Ω_d, f is a function and φ a measure on Ω_d. Equation (1.6) may be read in two ways: Either φ is given, then (1.6) is a functional equation for f, or f is given and φ is to be determined.

We shall now state a theorem which gives necessary and sufficient conditions for a pair f, φ in order that (1.6) be true. From these conditions the uniqueness theorems 1.1—1.5, and some others, can be immediately deduced via some elementary computations.