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are then seen from (10.9), (10.10) and (10.11) to satisfy

$$\left. \begin{aligned} \| Q_j' \| &\leq 1, \\ \text{sp } (Q_j') &\subseteq [A_j], \\ \left| \int v D_{A_j} Q_j' d\lambda_G \right| &\geq (1 - 3v^{-\frac{1}{2}}) \| D_{A_j} \|_1 \end{aligned} \right\} \quad (10.13)$$

provided v is chosen $\geq 9 \| D_{A_j} \|_1^{-1}$. In view of (7.6), we may choose the integer $v \geq \max_j (36, 9 \| D_{A_j} \|_1^{-1})$. Then (10.13) shows that there are unimodular complex numbers ξ_j such that the $Q_j = \xi_j Q_j'$ satisfy (7.7).

APPENDIX

Rudin-Shapiro sequences

A.1 NOTATIONS AND DEFINITIONS. As hitherto, all topological groups G are assumed to be Hausdorff; and, for any locally compact group G , λ_G will denote a selected left Haar measure, with respect to which the Lebesgue spaces $L^p(G)$ are to be formed. $C_c(G)$ denotes the set of complex-valued continuous functions on G having compact supports.

If X and Y are topological groups, $\text{Hom } (X, Y)$ denotes the set of continuous homomorphisms of X into Y .

Suppose henceforth G to be locally compact. As in 5.1, if $k \in C_c(G)$, T_k will denote the convolution operator

$$f \mapsto f * k$$

with domain $C_c(G)$ and range in $C_c(G)$; and $\| k \|_{p,q}$ will denote the (p, q) -norm of this operator, i.e., the smallest real number $m \geq 0$ such that

$$\| f * k \|_q \leq m \| f \|_p \quad (f \in C_c(G)).$$

It is well-known that, if G is Abelian, $\| k \|_{2,2}$ is equal to

$$\| \hat{k} \|_\infty = \sup_{\gamma \in \Gamma} | \hat{k}(\gamma) |,$$

where Γ is the character group of G and \hat{k} is the Fourier transform of k . (Something similar is true whenever G is compact, but we shall not use this.)

U -RS-sequences on G are as defined in 5.4.

In A.2-A.4 we are concerned with conditions on G sufficient to ensure the possibility of constructing U -RS-sequences on G for certain choices of U . In A.5 we use Rudin-Shapiro sequences on infinite compact Abelian groups to support statements made in 7.5.

A.2 THE ABELIAN CASE. If G is Abelian and nondiscrete, the methods of § 2 of [5] show how to construct (reasonably explicitly) a U -RS-sequence (h_n) on G for any preassigned nonvoid open $U \subseteq G$; see also [7], (37.19.b). In addition, we may assume that each \hat{h}_n is integrable on Γ , the character group of G . [To see this, let V be a compact neighbourhood of the origin of G and let W be a nonvoid subset of U such that $V + W \subseteq U$. Let $\{u_i\}$ be an approximate identity on G comprised of functions in $C_c(G)$ with supports in V and Fourier transforms in $L^1(\Gamma)$. Finally, let (k_n) be a W -RS-sequence; then for each $n \in N$ we may select i_n so that $(k_n * u_{i_n})$ is a U -RS-sequence with the further property that $(k_n * u_{i_n})^\wedge = \hat{k}_n \hat{u}_{i_n} \in L^1(\Gamma)$, as required.] We take this construction for granted (but see A.5 below) and use it to show how to construct U -RS-sequences on certain non-Abelian groups G . The basis of the extension is a simple technique of passage from a quotient group to the original, the crucial step being A.3.2 below.

A.3 THE NOT-NECESSARILY ABELIAN CASE.

A.3.1 Assume here that K is a compact normal subgroup of G . Let λ_K be normalised so that $\lambda_K(K) = 1$; and let $\pi : x \mapsto \bar{x}$ denote the natural mapping of G onto G/K .

If $f \in C_c(G)$, the function f' on G/K defined by

$$f'(\bar{x}) = \int_K f(xt) d\lambda_K(t) \quad (\text{A.1})$$

belongs to $C_c(G/K)$; cf. [7], (15.21). If $g \in C_c(G/K)$, $g \circ \pi \in C_c(G)$ and

$$(g \circ \pi)' = g. \quad (\text{A.2})$$

If τ_a denotes left-translation by amount a , it is verifiable that $(\tau_a f)' = \tau_{\bar{a}} f'$. From this it follows that the disposable factors in λ_G and $\lambda_{G/K}$ can be mutually adjusted so that

$$\int_G f d\lambda_G = \int_{G/K} f' d\lambda_{G/K} \quad (\text{A.3})$$

for $f \in C_c(G)$. Using (A.3), a direct calculation confirms that

$$(f * (k \circ \pi))' = f' * k \quad (\text{A.4})$$

whenever $f \in C_c(G)$ and $k \in C_c(G/K)$.

Another consequence of (A.3) is that for $1 \leq p \leq \infty$

$$\|f\|_p \geq \|f'\|_p \quad (\text{A.5})$$

for every $f \in C_c(G)$; and that for $0 < p \leq \infty$

$$\|f\|_p = \|f'\|_p \quad (\text{A.6})$$

for every $f \in C_c(G;K)$, the set of $f \in C_c(G)$ which are constant on cosets modulo K .

A.3.2 Let $k \in C_c(G/K)$. Then

$$\|k \circ \pi\|_{p,q} \leq \|k\|_{p,q}. \quad (\text{A.7})$$

PROOF. For $f \in C_c(G)$, $f * (k \circ \pi) \in C_c(G;K)$ and (A.6) gives

$$\|f * (k \circ \pi)\|_q = \|(f * (k \circ \pi))'\|_q,$$

which by (A.4)

$$\begin{aligned} &= \|f' * k\|_q \\ &\leq \|f'\|_p \|k\|_{p,q} \\ &\leq \|f\|_p \|k\|_{p,q}, \end{aligned}$$

the last step by (A.5). Whence (A.7).

A.3.3 If (h_n) is a V -RS-sequence on G/K and $U = \pi^{-1}(V)$, then $(h_n \circ \pi)$ is a U -RS-sequence on G .

PROOF. In view of A.3.2 it suffices to note that

$$\begin{aligned} \text{supp } (h_n \circ \pi) &= \pi^{-1}(\text{supp } h_n) \\ &\subseteq \pi^{-1}(V), \end{aligned}$$

$$\begin{aligned} \|h_n \circ \pi\|_\infty &= \|h_n\|_\infty, \\ \|h_n \circ \pi\|_2 &= \|h_n\|_2, \end{aligned}$$

the last two because of (A.6) and (A.2).

A.3.4 Suppose that K is a compact normal subgroup of G and that one can construct V -RS-sequences on G/K for any given nonvoid open $V \subseteq G/K$. Then one can construct U -RS-sequences on G for any given open subset U of G which contains K .

PROOF. Apply A.3.3, taking a nonvoid open subset W of G such that $KW \subseteq U$, and noting that $V = \pi(W)$ is then nonvoid and open in G/K and that $\pi^{-1}(V) = KW \subseteq U$.

A.3.5 Let $\delta(G)$ be the closure in G of the derived (= commutator) subgroup of G , and suppose that $\delta(G)$ is compact and nonopen in G . Then one can construct U -RS-sequences on G for any given open subset U of G containing $\delta(G)$. (Note that, since $\delta(G)$ is a closed subgroup of G , it is nonopen in G if and only if it has empty interior, or if and only if it is locally null for λ_G .)

PROOF. This follows from A.2 and A.3.4 because: $\delta(G)$ is in any case a normal subgroup of G such that $G/\delta(G)$ is LCA [see [7], (5.22), (5.26), (23.8)]; and $\delta(G)$ is nonopen in G if and only if $G/\delta(G)$ is nondiscrete ([7], (5.21)).

A.3.6 The hypotheses of A.3.5 are satisfied in any one of the following cases (all groups being assumed Hausdorff and locally compact):

(i) $G = G_1 \times G_2$, where $\delta(G_1)$ and $\delta(G_2)$ are compact and $\delta(G_1)$ is nonopen in G_1 (hence in particular if $G = A \times B$, where A is nondiscrete Abelian and $\delta(B)$ is compact);

(ii) $\delta(G)$ is compact and there exists an open connected subset W of G such that $e \in W \not\subseteq \delta(G)$ (hence in particular if G is compact and connected and $\delta(G) \neq G$);

(iii) $\delta(G)$ is compact and, for some Abelian A , some $\varphi \in \text{Hom}(G, A)$ and some connected open subset W of G , we have $e \in W$ and $\varphi|_W$ non-constant (hence in particular if G is compact and connected and $\text{Hom}(G, A)$ is nontrivial);

(iv) $G = \varphi(H)$, where $\varphi \in \text{Hom}(G, H)$ is such that $\text{Ker } \varphi$ is locally countable (that is, such that $\text{Ker } \varphi$ intersects each compact set in a countable set), and where $\delta(H)$ is compact and nonopen in H .

PROOF. (i) It is evident that $\delta(G) \subseteq \delta(G_1) \times \delta(G_2)$, which shows that $\delta(G)$ is compact and nonopen in G [if it were open, $\delta(G_1) = \text{pr}_{G_1}(\delta(G_1) \times \delta(G_2))$ would have interior points].

(ii) Were $\delta(G)$ to be open in G , W would be a disjoint union of $W \cap \delta(G)$ and $W \cap (G \setminus \delta(G))$, each relatively open in W . Since

$e \in W \cap \delta(G)$, connectedness of W would imply that $W \cap (G \setminus \delta(G)) = \emptyset$, i.e., $W \subseteq \delta(G)$, a contradiction.

(iii) $\text{Ker } \varphi$ is a closed subgroup of G containing $\delta(G)$; since $W \not\subseteq \text{Ker } \varphi$, it follows that $W \not\subseteq \delta(G)$. Now use (ii).

(iv) Clearly,

$$\delta(G) \subseteq \overline{\varphi(\delta(H))} = \varphi(\delta(H))$$

is compact. Suppose $\delta(G)$ were open in G . Then $\varphi(\delta(H))$ has interior points, and the same would be true of

$$\varphi^{-1}(\varphi(\delta(H))) = S\delta(H),$$

where $S = \text{Ker } \varphi$. So there would exist a compact neighbourhood V of the identity in H such that

$$V \subseteq S\delta(H)$$

and so

$$V = V \cap (S\delta(H)).$$

But, if $y \in V \cap (S\delta(H))$, $y = sz$ for some $s \in S$ and $z \in \delta(H)$, hence $s = yz^{-1} \in V\delta(H)^{-1}$, and so $s \in (V\delta(H)^{-1}) \cap S$, which is countable by hypothesis, say $\{s_n : n \in N\}$. But then

$$y \in \bigcup_{n \in N} s_n \delta(H).$$

Thus

$$V = V \cap (S\delta(H)) \subseteq \bigcup_{n \in N} s_n \delta(H)$$

and so, since $\lambda_H(\delta(H)) = 0$,

$$0 < \lambda_H(V) \leq \sum_{n \in N} \lambda_H(\delta(H)) = 0,$$

a contradiction.

A.3.7 REMARKS. (i) A.3.6 (iii) suffices to show that any finite-dimensional unitary group $U(n)$ satisfies the hypotheses of A.3.5. [For $U(n)$ is compact and connected (see [7], (7.15)); and we may apply A.3.6 (iii) with $A = T$, the circle group, and $\varphi = \det$.]

On the other hand, it is easy to see (cf. A.3.6 (i) and its proof) that if $G = \prod_{i \in I} G_i$, where the G_i are compact and at least one of them satisfies the hypothesis of A.3.5, then G satisfies the said hypotheses.

So every product of unitary groups satisfies the hypotheses of A.3.5.

(ii) The hypotheses of A.3.5 are also satisfied if $G = G_1 \oplus G_2$, the semidirect product of G_1 and G_2 (see [7], (2.6) and (6.20)), provided G_1 is compact and $\delta(G_2)$ is compact and nonopen in G_2 (hence in particular if $G = A \oplus B$, where A is compact and B is nondiscrete and Abelian). In fact, $\delta(G) \subseteq G_1 \times \delta(G_2)$ and the proof proceeds as for A.3.6 (i).

A.4 THE OPERATORS $f \mapsto k * f$. Retaining the notations introduced in A.3, it turns out that (cf. (A.4))

$$((k \circ \pi) * f)' = k * f^{\vee\vee} \quad (\text{A.8})$$

for every $f \in C_c(G)$ and $k \in C_c(G/K)$, where, for any function g with domain a group X , \check{g} denotes the function $x \mapsto g(x^{-1})$ with domain X . As a consequence, the results of A.3 have direct analogues for the operator $f \mapsto k * f$, provided G/K is unimodular, which is so if and only if G is unimodular.

A.5 CONCERNING 7.5.

A.5.1 Throughout A.5 we suppose G to be infinite compact Abelian. Let Γ_0 be any infinite subsemigroup of the character group Γ of G ; $0 \in \Gamma_0$. The construction described in § 2 of [5] may be employed to produce t.p.s. f_n ($n \in N$) on G which, together with their spectra S_n , satisfy the conditions:

$$\left. \begin{aligned} S_0 &= \{0\}, S_n \subseteq \Gamma_0, |S_n| = 2^n \\ B2^{n/2} &\leq \|f_n\|_s \leq A2^{n/2} \quad (1 \leq s \leq \infty), \\ \|f_n\|_{2,2} &= \|\hat{f}_n\|_\infty \leq 1, \\ \hat{f}_n &= \varphi \text{ on } S_n, 0 \text{ on } \Gamma \setminus S_n, \end{aligned} \right\} \quad (\text{A.9})$$

where A and B are positive absolute constants and φ is a function on Γ with $\text{Ran } \varphi \subseteq \{-1, 0, 1\}$ and $|\varphi(\gamma)| = 1$ if and only if $\gamma \in S_n$. (When $G = T$, these f_n are virtually the original Rudin-Shapiro t.p.s. In the terminology adopted in 5.4 above the $h_n = 2^{-n/2} f_n$ constitute a G -RS-sequence on G .)

If we now choose $\alpha_n \in \Gamma$ inductively so that, on writing $F_n = \alpha_n + S_n$, we have

$$\alpha_{n+1} \in \Gamma_0 \setminus [(F_0 \cup \dots \cup F_n) - S_{n+1}],$$

then

$$\left. \begin{aligned} |F_n| &= |S_n| = 2^n, F_n \subseteq \Gamma_0, \\ F_n \cap F_m &= \emptyset \text{ if } m \neq n, \end{aligned} \right\} \quad (\text{A.10})$$

and the t.p.s

$$w_n = 2^{-n/2} \alpha_n f_n \quad (\text{A.11})$$

satisfy the relations

$$\left. \begin{aligned} \|w_n\|_\infty &\leq A, \hat{w}_n = 2^{-n/2} \varphi_n, \\ \text{Ran } \varphi_n &\subseteq \{-1, 0, 1\}, |\varphi_n(\gamma)| = 1 \text{ if and only if } \gamma \in F_n. \end{aligned} \right\} \quad (\text{A.12})$$

From (A.10) and (A.12) it follows that at least one of the sets $A_n = \varphi_n^{-1}(\{1\})$, $B_n = \varphi_n^{-1}(\{-1\})$ has not fewer than 2^{n-1} elements. Define $\varepsilon_n = 1$, $C_n = A_n$ if $|A_n| \geq 2^{n-1}$ and $\varepsilon_n = -1$, $C_n = B_n$ if $|A_n| < 2^{n-1}$. Then

$$\left. \begin{aligned} (\varepsilon_n w_n)^\wedge(\gamma) &= 2^{-n/2} \text{ if } \gamma \in C_n, \\ C_n &\subseteq F_n, |C_n| \geq 2^{n-1}. \end{aligned} \right\} \quad (\text{A.13})$$

A.5.2 In terms of the construction given in A.5.1, it is possible to write down any number of continuous functions f on G and sequences (Δ_j) of finite subsets of Γ_0 such that

$$\left. \begin{aligned} \Delta_j &\subseteq \Delta_{j+1}, \\ \text{sp}(f) &\subseteq \Gamma_0, \\ S_{\Delta_j} f(0) &\text{ is real and } \lim_{j \rightarrow \infty} S_{\Delta_j} f(0) = \infty, \\ \sum_{\gamma \in \Gamma} |\hat{f}(\gamma)| &= \infty; \end{aligned} \right\} \quad (\text{A.14})$$

cf. the statements made in 7.5.

Indeed, if $(c_n)_{n=0}^\infty$ is a sequence of real numbers satisfying

$$c_n \geq 0, \sum_{n=0}^\infty c_n < \infty, \sum_{n=0}^\infty 2^{n/2} c_n = \infty, \quad (\text{A.15})$$

and if

$$\Delta_j = C_0 \cup \dots \cup C_j, \quad (\text{A.16})$$

it suffices to take

$$f = \sum_{n=0}^\infty c_n \varepsilon_n w_n, \quad (\text{A.17})$$

(A.14) being then a simple consequence of (A.12) and (A.13).

However, it is a consequence of the choice of the γ_n and α_n and of (A.12) [on evaluating the Fourier series of w_n at 0] that $||A_n| - |B_n|| \leq 2^{n/2}$, which implies that C_n contains only about one half the elements of F_n , so that $\bigcup_{j=1}^{\infty} \Delta_j$ falls far short of exhausting Γ_0 . In particular, (Δ_j) is not a convergence grouping of the sort described in § 7.

A.5.3 Two further consequences of the construction in A.5.1 are perhaps worth mentioning in passing.

(i) For any complex-valued sequence $(c_n)_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} |c_n| < \infty, \quad (\text{A.18})$$

the formula

$$g = \sum_{n=1}^{\infty} c_n w_n \quad (\text{A.19})$$

yields a continuous function $g \in C(G)$. It is easy to specify choices of (c_n) in accord with (A.18), and of nonnegative functions η on Γ such that

$$\lim_{\gamma \rightarrow \infty} \eta(\gamma) = 0, \quad (\text{A.20})$$

for which

$$\sum_{\gamma \in \Gamma} |\hat{g}(\gamma)|^{2-2\eta(\gamma)} = \infty. \quad (\text{A.21})$$

One might, for example, take $c_n = n^{-2}$ and $\eta(\gamma) = 6n^{-1} \log n$ for $\gamma \in F_n$ ($n = 1, 2, \dots$) and $\eta(\gamma) = 0$ for $\gamma \in \Gamma \setminus F$, where $F = \bigcup_{n=1}^{\infty} F_n$.

This is an analogue of a well-known result of Banach for the case $G = T$; it provides numerous reasonably constructive counter-examples to conjectural improvements of the Hausdorff-Young theorem.

(ii) Take (c_n) , η and g as in (i) immediately above. Let ψ be any nonnegative function on Γ which is bounded away from zero on F . Let further θ be any complex-valued function on Γ such that

$$\theta(\gamma) = \psi(\gamma) |\hat{g}(\gamma)|^{1-2\eta(\gamma)} \cdot \text{sgn } \hat{g}(\gamma) \quad \text{for } \gamma \in F. \quad (\text{A.22})$$

Then (A.21), (A.22) and Bochner's theorem combine to show that θ is

not a Fourier-Stieltjes transform. Yet, if ψ is bounded, and if we define $\theta(\gamma) = 0$ for $\gamma \in \Gamma \setminus F$, (A.20) and the fact that $g \in C(G)$ ensure that

$$\theta \in \bigcap_{r>2} l^r(\Gamma). \quad (\text{A.23})$$

We thus obtain explicit examples of functions θ satisfying (A.23) which are not Fourier-Stieltjes transforms.

Note that, if every c_n is real and nonzero, an (unbounded) ψ can be chosen so as to make $\text{Ran } \theta = \{-1, 1\}$; this yields explicit examples of ± 1 -valued functions θ which are not Fourier-Stieltjes transforms. (These are, of course, also obtainable by starting with functions $\text{sgn } \hat{h}$, where $h \in C(G)$, \hat{h} is real-valued and $\hat{h} \notin l^1(\Gamma)$.)

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