**Zeitschrift:** L'Enseignement Mathématique

**Herausgeber:** Commission Internationale de l'Enseignement Mathématique

**Band:** 16 (1970)

**Heft:** 1: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** NAIVELY CONSTRUCTIVE APPROACH TO BOUNDEDNESS PRINCIPLES, WITH APPLICATIONS TO HARMONIC ANALYSIS

**Autor:** Edwards, R. E. / Price, J. F.

**Anhang:** Appendix Rudin-Shapiro sequences

**DOI:** https://doi.org/10.5169/seals-43866

are then seen from (10.9), (10.10) and (10.11) to satisfy

$$\left.\begin{aligned} \| Q_j' \| &\leqq 1, \\ \mathrm{sp}\,(Q_j') &\subseteq [\Delta_j], \\ \left| \int v\, D_{\Delta_j}\, Q_j'\, d\lambda_G \right| &\geqq (1 - 3v^{-\frac{1}{2}})\, \| D_{\Delta_j} \|_1 \end{aligned}\right\} \tag{10.13}$$

provided $v$ is chosen $\geqq 9 \| D_{\Delta_j} \|_1^{-1}$. In view of (7.6), we may choose the integer $v \geqq \max_j (36, 9 \| D_{\Delta_j} \|_1^{-1})$. Then (10.13) shows that there are unimodular complex numbers $\xi_j$ such that the $Q_j = \xi_j Q_j'$ satisfy (7.7).

## APPENDIX

### *Rudin-Shapiro sequences*

A.1 NOTATIONS AND DEFINITIONS. As hitherto, all topological groups $G$ are assumed to be Hausdorff; and, for any locally compact group $G$, $\lambda_G$ will denote a selected left Haar measure, with respect to which the Lebesgue spaces $L^p(G)$ are to be formed. $C_c(G)$ denotes the set of complex-valued continuous functions on $G$ having compact supports.

If $X$ and $Y$ are topological groups, Hom $(X, Y)$ denotes the set of continuous homomorphisms of $X$ into $Y$.

Suppose henceforth $G$ to be locally compact. As in 5.1, if $k \in C_c(G)$, $T_k$ will denote the convolution operator

$$f \mapsto f * k$$

with domain $C_c(G)$ and range in $C_c(G)$; and $\| k \|_{p,q}$ will denote the $(p, q)$-norm of this operator, i.e., the smallest real number $m \geqq 0$ such that

$$\| f * k \|_q \leqq m \| f \|_p \quad (f \in C_c(G)).$$

It is well-known that, if $G$ is Abelian, $\| k \|_{2,2}$ is equal to

$$\| \hat{k} \|_\infty = \sup_{\gamma \in \Gamma} | \hat{k}(\gamma) |,$$

where $\Gamma$ is the character group of $G$ and $\hat{k}$ is the Fourier transform of $k$. (Something similar is true whenever $G$ is compact, but we shall not use this.)

$U$-RS-sequences on $G$ are as defined in 5.4.

In A.2-A.4 we are concerned with conditions on $G$ sufficient to ensure the possibility of constructing $U$-RS-sequences on $G$ for certain choices of $U$. In A.5 we use Rudin-Shapiro sequences on infinite compact Abelian groups to support statements made in 7.5.

A.2 The Abelian case. If $G$ is Abelian and nondiscrete, the methods of § 2 of [5] show how to construct (reasonably explicitly) a $U$-RS-sequence $(h_n)$ on $G$ for any preassigned nonvoid open $U \subseteq G$; see also [7], (37.19.b). In addition, we may assume that each $\hat{h}_n$ is integrable on $\Gamma$, the character group of $G$. [To see this, let $V$ be a compact neighbourhood of the origin of $G$ and let $W$ be a nonvoid subset of $U$ such that $V + W \subseteq U$. Let $\{u_i\}$ be an approximate identity on $G$ comprised of functions in $C_c(G)$ with supports in $V$ and Fourier transforms in $L^1(\Gamma)$. Finally, let $(k_n)$ be a $W$-RS-sequence; then for each $n \in N$ we may select $i_n$ so that $(k_n * u_{i_n})$ is a $U$-RS-sequence with the further property that $(k_n * u_{i_n})^\wedge = \hat{k}_n \hat{u}_{i_n} \in L^1(\Gamma)$, as required.] We take this construction for granted (but see A.5 below) and use it to show how to construct $U$-RS-sequences on certain non-Abelian groups $G$. The basis of the extension is a simple technique of passage from a quotient group to the original, the crucial step being A.3.2 below.

A.3 The not-necessarily Abelian case.

A.3.1 Assume here that $K$ is a compact normal subgroup of $G$. Let $\lambda_K$ be normalised so that $\lambda_K(K) = 1$; and let $\pi : x \mapsto \bar{x}$ denote the natural mapping of $G$ onto $G/K$.

If $f \in C_c(G)$, the function $f'$ on $G/K$ defined by

$$f'(\bar{x}) = \int_K f(xt)\, d\lambda_K(t) \tag{A.1}$$

belongs to $C_c(G/K)$; cf. [7], (15.21). If $g \in C_c(G/K)$, $g \circ \pi \in C_c(G)$ and

$$(g \circ \pi)' = g. \tag{A.2}$$

If $\tau_a$ denotes left-translation by amount $a$, it is verifiable that $(\tau_a f)' = \tau_{\bar{a}} f'$. From this it follows that the disposable factors in $\lambda_G$ and $\lambda_{G/K}$ can be mutually adjusted so that

$$\int_G f\, d\lambda_G = \int_{G/K} f'\, d\lambda_{G/K} \tag{A.3}$$

for $f \in C_c(G)$. Using (A.3), a direct calculation confirms that

$$(f * (k \circ \pi))' = f' * k \tag{A.4}$$

whenever $f \in C_c(G)$ and $k \in C_c(G/K)$.

Another consequence of (A.3) is that for $1 \leqq p \leqq \infty$

$$\|f\|_p \geqq \|f'\|_p \tag{A.5}$$

for every $f \in C_c(G)$; and that for $0 < p \leqq \infty$

$$\|f\|_p = \|f'\|_p \tag{A.6}$$

for every $f \in C_c(G;K)$, the set of $f \in C_c(G)$ which are constant on cosets modulo $K$.

A.3.2 Let $k \in C_c(G/K)$. Then

$$\| k \circ \pi \|_{p,q} \leqq \| k \|_{p,q}. \tag{A.7}$$

Proof. For $f \in C_c(G)$, $f * (k \circ \pi) \in C_c(G;K)$ and (A.6) gives

$$\|f * (k \circ \pi)\|_q = \|(f * (k \circ \pi))'\|_q,$$

which by (A.4)

$$\begin{aligned} &= \|f' * k\|_q \\ &\leqq \|f'\|_p \|k\|_{p,q} \\ &\leqq \|f\|_p \|k\|_{p,q}, \end{aligned}$$

the last step by (A.5). Whence (A.7).

A.3.3 If $(h_n)$ is a $V$-RS-sequence on $G/K$ and $U = \pi^{-1}(V)$, then $(h_n \circ \pi)$ is a $U$-RS-sequence on $G$.

Proof. In view of A.3.2 it suffices to note that

$$\begin{aligned} \operatorname{supp}(h_n \circ \pi) &= \pi^{-1}(\operatorname{supp} h_n) \\ &\subseteq \pi^{-1}(V), \\ \| h_n \circ \pi \|_\infty &= \| h_n \|_\infty, \\ \| h_n \circ \pi \|_2 &= \| h_n \|_2, \end{aligned}$$

the last two because of (A.6) and (A.2).

A.3.4 Suppose that $K$ is a compact normal subgroup of $G$ and that one can construct $V$-RS-sequences on $G/K$ for any given nonvoid open $V \subseteq G/K$. Then one can construct $U$-RS-sequences on $G$ for any given open subset $U$ of $G$ which contains $K$.

PROOF. Apply A.3.3, taking a nonvoid open subset $W$ of $G$ such that $KW \subseteq U$, and noting that $V = \pi(W)$ is then nonvoid and open in $G/K$ and that $\pi^{-1}(V) = KW \subseteq U$.

A.3.5 Let $\delta(G)$ be the closure in $G$ of the derived (= commutator) subgroup of $G$, and suppose that $\delta(G)$ is compact and nonopen in $G$. Then one can construct $U$-RS-sequences on $G$ for any given open subset $U$ of $G$ containing $\delta(G)$. (Note that, since $\delta(G)$ is a closed subgroup of $G$, it is nonopen in $G$ if and only if it has empty interior, or if and only if it is locally null for $\lambda_G$.)

PROOF. This follows from A.2 and A.3.4 because:
$\delta(G)$ is in any case a normal subgroup of $G$ such that $G/\delta(G)$ is LCA [see [7], (5.22), (5.26), (23.8)]; and $\delta(G)$ is nonopen in $G$ if and only if $G/\delta(G)$ is nondiscrete ([7], (5.21)).

A.3.6 The hypotheses of A.3.5 are satisfied in any one of the following cases (all groups being assumed Hausdorff and locally compact):

(i) $G = G_1 \times G_2$, where $\delta(G_1)$ and $\delta(G_2)$ are compact and $\delta(G_1)$ is nonopen in $G_1$ (hence in particular if $G = A \times B$, where $A$ is nondiscrete Abelian and $\delta(B)$ is compact);

(ii) $\delta(G)$ is compact and there exists an open connected subset $W$ of $G$ such that $e \in W \nsubseteq \delta(G)$ (hence in particular if $G$ is compact and connected and $\delta(G) \neq G$);

(iii) $\delta(G)$ is compact and, for some Abelian $A$, some $\varphi \in \mathrm{Hom}(G, A)$ and some connected open subset $W$ of $G$, we have $e \in W$ and $\varphi \mid W$ nonconstant (hence in particular if $G$ is compact and connected and $\mathrm{Hom}(G, A)$ is nontrivial);

(iv) $G = \varphi(H)$, where $\varphi \in \mathrm{Hom}(G, H)$ is such that $\mathrm{Ker}\,\varphi$ is locally countable (that is, such that $\mathrm{Ker}\,\varphi$ intersects each compact set in a countable set), and where $\delta(H)$ is compact and nonopen in $H$.

PROOF. (i) It is evident that $\delta(G) \subseteq \delta(G_1) \times \delta(G_2)$, which shows that $\delta(G)$ is compact and nonopen in $G$ [if it were open, $\delta(G_1) = pr_{G_1}(\delta(G_1) \times \delta(G_2))$ would have interior points].

(ii) Were $\delta(G)$ to be open in $G$, $W$ would be a disjoint union of $W \cap \delta(G)$ and $W \cap (G \setminus \delta(G))$, each relatively open in $W$. Since

$e \in W \cap \delta(G)$, connectedness of $W$ would imply that $W \cap (G \backslash \delta(G)) = \varnothing$, i.e., $W \subseteq \delta(G)$, a contradiction.

(iii) Ker $\varphi$ is a closed subgroup of $G$ containing $\delta(G)$; since $W \nsubseteq \text{Ker}\,\varphi$, it follows that $W \nsubseteq \delta(G)$. Now use (ii).

(iv) Clearly,

$$\delta(G) \subseteq \overline{\varphi\,(\delta(H))} = \varphi\,(\delta(H))$$

is compact. Suppose $\delta(G)$ were open in $G$. Then $\varphi(\delta(H))$ has interior points, and the same would be true of

$$\varphi^{-1}\,(\varphi\,(\delta(H))) = S\,\delta(H),$$

where $S = \text{Ker}\,\varphi$. So there would exist a compact neighbourhood $V$ of the identity in $H$ such that

$$V \subseteq S\delta(H)$$

and so

$$V = V \cap (S\delta(H)).$$

But, if $y \in V \cap (S\delta(H))$, $y = sz$ for some $s \in S$ and $z \in \delta(H)$, hence $s = yz^{-1} \in V\delta(H)^{-1}$, and so $s \in (V\delta(H)^{-1}) \cap S$, which is countable by hypothesis, say $\{s_n : n \in N\}$. But then

$$y \in \bigcup_{n \in N} s_n\,\delta(H).$$

Thus

$$V = V \cap (S\delta(H)) \subseteq \bigcup_{n \in N} s_n\,\delta(H)$$

and so, since $\lambda_H(\delta(H)) = 0$,

$$0 < \lambda_H\,(V) \leqq \sum_{n \in N} \lambda_H\,(\delta(H)) = 0,$$

a contradiction.

A.3.7 Remarks. (i) A.3.6 (iii) suffices to show that any finite-dimensional unitary group $U(n)$ satisfies the hypotheses of A.3.5. [For $U(n)$ is compact and connected (see [7], (7.15)); and we may apply A.3.6 (iii) with $A = T$, the circle group, and $\varphi = \det$.]

On the other hand, it is easy to see (cf. A.3.6 (i) and its proof) that if $G = \prod_{i \in I} G_i$, where the $G_i$ are compact and at least one of them satisfies the hypothesis of A.3.5, then $G$ satisfies the said hypotheses.

So every product of unitary groups satisfies the hypotheses of A.3.5.

(ii) The hypotheses of A.3.5 are also satisfied if $G = G_1 \circledS G_2$, the semidirect product of $G_1$ and $G_2$ (see [7], (2.6) and (6.20)), provided $G_1$ is compact and $\delta(G_2)$ is compact and nonopen in $G_2$ (hence in particular if $G = A \circledS B$, where $A$ is compact and $B$ is nondiscrete and Abelian). In fact, $\delta(G) \subseteq G_1 \times \delta(G_2)$ and the proof proceeds as for A.3.6 (i).

A.4 THE OPERATORS $f \mapsto k * f$. Retaining the notations introduced in A.3, it turns out that (cf. (A.4))

$$((k \circ \pi) * f)' = k * f^{\vee \prime \vee} \tag{A.8}$$

for every $f \in C_c(G)$ and $k \in C_c(G/K)$, where, for any function $g$ with domain a group $X$, $\check{g}$ denotes the function $x \mapsto g(x^{-1})$ with domain $X$. As a consequence, the results of A.3 have direct analogues for the operator $f \mapsto k * f$, provided $G/K$ is unimodular, which is so if and only if $G$ is unimodular.

A.5 CONCERNING 7.5.

A.5.1 Throughout A.5 we suppose $G$ to be infinite compact Abelian. Let $\Gamma_0$ be any infinite subsemigroup of the character group $\Gamma$ of $G$; $0 \in \Gamma_0$. The construction described in § 2 of [5] may be employed to produce t.p.s $f_n$ $(n \in N)$ on $G$ which, together with their spectra $S_n$, satisfy the conditions:

$$\left.\begin{aligned} &S_0 = \{0\},\ S_n \subseteq \Gamma_0,\ |S_n| = 2^n \\ &B2^{n/2} \leqq \|f_n\|_s \leqq A2^{n/2} \quad (1 \leqq s \leqq \infty), \\ &\|f_n\|_{2,2} = \|\hat{f}_n\|_\infty \leqq 1, \\ &\hat{f}_n = \varphi \text{ on } S_n,\ 0 \text{ on } \Gamma \setminus S_n, \end{aligned}\right\} \tag{A.9}$$

where $A$ and $B$ are positive absolute constants and $\varphi$ is a function on $\Gamma$ with $\operatorname{Ran}\varphi \subseteq \{-1, 0, 1\}$ and $|\varphi(\gamma)| = 1$ if and only if $\gamma \in S_n$. (When $G = T$, these $f_n$ are virtually the original Rudin-Shapiro t.p.s. In the terminology adopted in 5.4 above the $h_n = 2^{-n/2} f_n$ constitute a $G$-RS-sequence on $G$.)

If we now choose $\alpha_n \in \Gamma$ inductively so that, on writing $F_n = \alpha_n + S_n$, we have

$$\alpha_{n+1} \in \Gamma_0 \setminus [(F_0 \cup \dots \cup F_n) - S_{n+1}],$$

then

$$\left.\begin{aligned} &|F_n| = |S_n| = 2^n,\ F_n \subseteq \Gamma_0, \\ &F_n \cap F_m = \varnothing \text{ if } m \neq n, \end{aligned}\right\} \tag{A.10}$$

and the t.p.s

$$w_n = 2^{-n/2}\alpha_n f_n \tag{A.11}$$

satisfy the relations

$$\left.\begin{aligned} &\| w_n \|_\infty \leqq A,\ \hat{w}_n = 2^{-n/2}\varphi_n, \\ \text{Ran } \varphi_n \subseteq \{-1, 0, 1\},\ &|\varphi_n(\gamma)| = 1 \quad \text{if and only if} \quad \gamma \in F_n. \end{aligned}\right\} \tag{A.12}$$

From (A.10) and (A.12) it follows that at least one of the sets $A_n = \varphi_n^{-1}(\{1\})$, $B_n = \varphi_n^{-1}(\{-1\})$ has not fewer than $2^{n-1}$ elements. Define $\varepsilon_n = 1$, $C_n = A_n$ if $|A_n| \geqq 2^{n-1}$ and $\varepsilon_n = -1$, $C_n = B_n$ if $|A_n| < 2^{n-1}$. Then

$$\left.\begin{aligned} (\varepsilon_n w_n)^\wedge(\gamma) = 2^{-n/2} \quad \text{if} \quad \gamma \in C_n, \\ C_n \subseteq F_n,\ |C_n| \geqq 2^{n-1}. \end{aligned}\right\} \tag{A.13}$$

A.5.2 In terms of the construction given in A.5.1, it is possible to write down any number of continuous functions $f$ on $G$ and sequences $(\Delta_j)$ of finite subsets of $\Gamma_0$ such that

$$\left.\begin{aligned} &\Delta_j \subseteq \Delta_{j+1}, \\ &\text{sp}(f) \subseteq \Gamma_0, \\ &S_{\Delta_j} f(0) \quad \text{is real and} \quad \lim_{j\to\infty} S_{\Delta_j} f(0) = \infty, \\ &\sum_{\gamma\in\Gamma} |\hat{f}(\gamma)| = \infty; \end{aligned}\right\} \tag{A.14}$$

cf. the statements made in 7.5.

Indeed, if $(c_n)_{n=0}^{\infty}$ is a sequence of real numbers satisfying

$$c_n \geqq 0,\ \sum_{n=0}^{\infty} c_n < \infty,\ \sum_{n=0}^{\infty} 2^{n/2} c_n = \infty, \tag{A.15}$$

and if

$$\Delta_j = C_0 \cup \ldots \cup C_j, \tag{A.16}$$

if suffices to take

$$f = \sum_{n=0}^{\infty} c_n \varepsilon_n w_n, \tag{A.17}$$

(A.14) being then a simple consequence of (A.12) and (A.13).

However, it is a consequence of the choice of the $\gamma_n$ and $\alpha_n$ and of (A.12) [on evaluating the Fourier series of $w_n$ at 0] that $\big| \, | A_n | - | B_n | \, \big| \leqq 2^{n/2}$, which implies that $C_n$ contains only about one half the elements of $F_n$, so that $\bigcup_{j=1}^{\infty} \Delta_j$ falls far short of exhausting $\Gamma_0$. In particular, $(\Delta_j)$ is not a convergence grouping of the sort described in § 7.

A.5.3 Two further consequences of the construction in A.5.1 are perhaps worth mentioning in passing.

(i) For any complex-valued sequence $(c_n)_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} | c_n | < \infty, \tag{A.18}$$

the formula

$$g = \sum_{n=1}^{\infty} c_n w_n \tag{A.19}$$

yields a continuous function $g \in C(G)$. It is easy to specify choices of $(c_n)$ in accord with (A.18), and of nonnegative functions $\eta$ on $\Gamma$ such that

$$\lim_{\gamma \to \infty} \eta(\gamma) = 0, \tag{A.20}$$

for which

$$\sum_{\gamma \in \Gamma} | \hat{g}(\gamma) |^{2 - 2\eta(\gamma)} = \infty. \tag{A.21}$$

One might, for example, take $c_n = n^{-2}$ and $\eta(\gamma) = 6n^{-1} \log n$ for $\gamma \in F_n$ $(n = 1, 2, \ldots)$ and $\eta(\gamma) = 0$ for $\gamma \in \Gamma \backslash F$, where $F = \bigcup_{n=1}^{\infty} F_n$.

This is an analogue of a well-known result of Banach for the case $G = T$; it provides numerous reasonably constructive counter-examples to conjectural improvements of the Hausdorff-Young theorem.

(ii) Take $(c_n)$, $\eta$ and $g$ as in (i) immediately above. Let $\psi$ be any nonnegative function on $\Gamma$ which is bounded away from zero on $F$. Let further $\theta$ be any complex-valued function on $\Gamma$ such that

$$\theta(\gamma) = \psi(\gamma) \, | \hat{g}(\gamma) |^{1 - 2\eta(\gamma)} \, . \operatorname{sgn} \hat{g}(\gamma) \quad \text{for} \quad \gamma \in F. \tag{A.22}$$

Then (A.21), (A.22) and Bochner's theorem combine to show that $\theta$ is not a Fourier-Stieltjes transform. Yet, if $\psi$ is bounded, and if we define $\theta(\gamma) = 0$ for $\gamma \in \Gamma \backslash F$, (A.20) and the fact that $g \in C(G)$ ensure that

$$\theta \in \bigcap_{r > 2} l^r(\Gamma). \tag{A.23}$$

We thus obtain explicit examples of functions $\theta$ satisfying (A.23) which are not Fourier-Stieltjes transforms.

Note that, if every $c_n$ is real and nonzero, an (unbounded) $\psi$ can be chosen so as to make Ran $\theta = \{-1, 1\}$; this yields explicit examples of $\pm$ 1-valued functions $\theta$ which are not Fourier-Stieltjes transforms. (These are, of course, also obtainable by starting with functions sgn $\hat{h}$, where $h \in C(G)$, $\hat{h}$ is real-valued and $\hat{h} \notin l^1(\Gamma)$.)

## BIBLIOGRAPHY


[1] Brainerd, B. and R. E. Edwards, Linear operators which commute with translations. Part 1: Representation theorems. *J. Australian Math. Soc.* 6 (1966), 289-327.

[2] Edwards, R. E., *Functional Analysis: Theory and Applications.* Holt, Rinehart and Winston, New York (1965).

[3] Edwards, R. E. and E. Hewitt, Pointwise limits for sequences of convolution operators. *Acta Math.* 113 (1965), 181-218.

[4] Figa-Talamanca, A. and G. I. Gaudry, Multipliers and sets of uniqueness of $L^p$. *Michigan Math. J.* 17 (1970), 179-191.

[5] Gaudry, G. I., Bad behaviour and inclusion results for multipliers of type $(p, q)$. *Pacific J. Math.* 35 (1970), 83-94.

[6] Gel'fand, I. M. and G. E. Shilov, *Generalized Functions, Volume 1.* Academic Press, New York (1964).

[7] Hewitt, E. and K. A. Ross, *Abstract Harmonic Analysis, Vols. I, II.* Springer-Verlag, Berlin (1963, 1970).

[8] Hewitt, E. and H. S. Zuckerman, On a theorem of P. J. Cohen and H. Davenport. *Proc. Amer. Math. Soc.* 14 (1963), 847-855.

[9] Price, J. F., Some strict inclusions between spaces of $L^p$-multipliers. To appear in *Trans. Amer. Math. Soc.* 152 (1970).

[10] Rudin, W., *Fourier Analysis on Groups.* Interscience, New York (1962).

[11] Zygmund, A., *Trigonometric Series, Vols. I & II.* Cambridge University Press (1959).

[12] Holland, Samuel S., Jr. A Hilbert space proof of the Banach-Steinhaus theorem. *Amer. Math. Monthly* 76 (1969), 40-41.

[13] Halmos, P. *A Hilbert space problem book.* Van Nostrand, Princeton, N.J. (1967).

[14] Hörmander, L., Estimates for translation invariant operators in $L^p$ spaces. *Acta Math.* 104 (1960), 93-140.

[15] Bourbaki, N., *Espaces Vectoriels Topologiques, Ch. I, II.* Hermann et Cie, Paris (1953).

[16] Brooks, J. K. and J. Mikusiński., On some theorems in functional analysis. *Bull. Acad. Pol. Sci.* (3) XVIII (1970), 151-155.

[17] Dunkl, C. F. and D. E. Ramirez, $L^p$ Multipliers on Compact Groups. To appear.



Department of Mathematics,
Institute of Advanced Studies,
Australian National University.


*(Reçu le 1 septembre 1970)*