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## § 10. *Concerning the polynomials* $Q_j$.

There is no difficulty in making fairly explicit the construction of t.p.s $Q_j$ of the type employed in 7.6.

For $p > 0$, $t \geqq 0$ define

$$h_p(t) = \begin{cases} 1 & \text{if } t \leqq p, \\ 2\left(1 - \dfrac{t}{2p}\right) & \text{if } p \leqq t \leqq 2p, \\ 0 & \text{if } t \geqq 2p. \end{cases} \tag{10.1}$$

For all complex $z$ define

$$f_p(z) = \begin{cases} 0 & \text{if } z = 0, \\ |z|^{-1} \bar{z} h_p(|z|) & \text{if } z \neq 0. \end{cases} \tag{10.2}$$

Write

$$\left.\begin{aligned} E_n(z) &= \pi^{-1} n \exp(-n|z|^2, \\ P_{n,k}(z) &= \pi^{-1} n \sum_{j=0}^{k} \frac{(-1)^j}{j!} (n|z|^2)^j \end{aligned}\right\} \tag{10.3}$$

Let $\mu$ denote Lebesgue measure on $C$ (identified with $R^2$ in the canonical fashion).

It is then routine to verify that

$$\left.\begin{aligned} &\| E_n * f_p \|_\infty \leqq \| f_p \|_\infty = 1, \\ &\lim_{n \to \infty} E_n * f_p = f_p \end{aligned}\right\} \tag{10.4}$$

uniformly on any compact set omitting 0. From this it follows that to every $p > 0$ and every positive integer $v$ correspond positive integers $\bar{n}(p, v)$, $\bar{k}(p, v)$ such that

$$\left.\begin{aligned} &\left| |z|^{-1} \bar{z} - f_p * P_{\bar{n}, \bar{k}}(z) \right| \leqq \frac{1}{v} \text{ for } \frac{1}{v} \leqq |z| \leqq p, \\ &|f_p * P_{\bar{n}, \bar{k}}(z)| \leqq 1 + \frac{1}{v} \text{ for } |z| \leqq p. \end{aligned}\right\} \tag{10.5}$$

Now

$$f_p * P_{\bar{n}, \bar{k}}(z) = q_{p,v}(z, \bar{z}), \tag{10.6}$$

where

$$q_{p,\nu}(X, Y) = \pi^{-1}\,\bar{n}(p,\nu) \sum_{j=0}^{\bar{k}(p,\nu)} \frac{(-\bar{n}(p,\nu))^j}{j!} \sum_{l=0}^{j} \sum_{m=0}^{j} \binom{j}{l}\binom{j}{m} X^l Y^m$$

$$(-1)^{l+m} \int \zeta^{j-l} \bar{\zeta}^{j-m} f_p(\zeta)\, d\mu(\zeta)$$

$$= \sum_{l,m=0}^{\bar{k}(p,\nu)} C_{p,\nu}(l, m)\, X^l Y^m. \tag{10.7}$$

It is easily verifiable that the $C_{p,\nu}(l, m)$ are real-valued.

If $\theta$ is a bounded measurable function on $G$ and

$$\overset{\circ}{Q}_{p,\nu} = q_{p,\nu}(\theta, \bar{\theta}),\ p \geqq \|\theta\|_\infty, \tag{10.8}$$

we have from (10.5)

$$\left. \begin{aligned} \left|\, |\theta|^{-1}\bar{\theta} - \overset{\circ}{Q}_{p,\nu} \right| &< \frac{1}{\nu} \text{ whenever } |\theta| \geqq \frac{1}{\nu}, \\ |\overset{\circ}{Q}_{p,\nu}| &\leqq 1 + \frac{1}{\nu} \text{ everywhere on } G. \end{aligned} \right\} \tag{10.9}$$

If $\theta$ is a t.p., then $\overset{\circ}{Q}_{p,\nu}$ is a t.p. and

$$\operatorname{sp}(\overset{\circ}{Q}_{p,\nu}) \subseteq [\operatorname{sp}(\theta)]. \tag{10.10}$$

From (10.9) we obtain

$$\left|\, |\theta| - \theta\, \overset{\circ}{Q}_{p,\nu} \right| \leqq \begin{cases} \nu^{-1}|\theta| \text{ whenever } |\theta| > \dfrac{1}{\nu}, \\ \left(2+\dfrac{1}{\nu}\right)|\theta| \text{ everywhere,} \end{cases}$$

whence it follows that, if $\theta \neq 0$,

$$\begin{aligned} \left| \int_G \theta\, \overset{\circ}{Q}_{p,\nu}\, d\lambda G \right| &\geqq (1-\nu^{-1})\,\|\theta\|_1 - \nu^{-1}(2+\nu^{-1}) \\ &\geqq (1-2\nu^{-\frac{1}{2}})\,\|\theta\|_1 \end{aligned} \tag{10.11}$$

provided $\nu \geqq 9\,\|\theta\|_1^{-2}$.

Taking $\theta = D_{\Delta_j}$ and $p_j \geqq \|D_{\Delta_j}\|$, the trigonometric polynomials

$$Q_j' = \left(1+\frac{1}{\nu}\right)^{-1} \overset{\circ}{Q}_{p_j,\nu} = \left(1+\frac{1}{\nu}\right)^{-1} q_{p_j,\nu}(D_{\Delta_j}, \bar{D}_{\Delta_j}) \tag{10.12}$$

are then seen from (10.9), (10.10) and (10.11) to satisfy

$$\left.\begin{aligned} \| Q_j' \| &\leqq 1, \\ \mathrm{sp}\,(Q_j') &\subseteq [\Delta_j], \\ \left| \int v\, D_{\Delta_j}\, Q_j'\, d\lambda_G \right| &\geqq (1 - 3v^{-\frac{1}{2}})\, \| D_{\Delta_j} \|_1 \end{aligned}\right\} \qquad (10.13)$$

provided $v$ is chosen $\geqq 9 \| D_{\Delta_j} \|_1^{-1}$. In view of (7.6), we may choose the integer $v \geqq \max_j (36, 9 \| D_{\Delta_j} \|_1^{-1})$. Then (10.13) shows that there are unimodular complex numbers $\xi_j$ such that the $Q_j = \xi_j Q_j'$ satisfy (7.7).

## Appendix

### *Rudin-Shapiro sequences*

A.1 Notations and definitions. As hitherto, all topological groups $G$ are assumed to be Hausdorff; and, for any locally compact group $G$, $\lambda_G$ will denote a selected left Haar measure, with respect to which the Lebesgue spaces $L^p(G)$ are to be formed. $C_c(G)$ denotes the set of complex-valued continuous functions on $G$ having compact supports.

If $X$ and $Y$ are topological groups, Hom $(X, Y)$ denotes the set of continuous homomorphisms of $X$ into $Y$.

Suppose henceforth $G$ to be locally compact. As in 5.1, if $k \in C_c(G)$, $T_k$ will denote the convolution operator

$$f \mapsto f * k$$

with domain $C_c(G)$ and range in $C_c(G)$; and $\| k \|_{p,q}$ will denote the $(p, q)$-norm of this operator, i.e., the smallest real number $m \geqq 0$ such that

$$\| f * k \|_q \leqq m \| f \|_p \quad (f \in C_c(G)).$$

It is well-known that, if $G$ is Abelian, $\| k \|_{2,2}$ is equal to

$$\| \hat{k} \|_\infty = \sup_{\gamma \in \Gamma} | \hat{k}(\gamma) |,$$

where $\Gamma$ is the character group of $G$ and $\hat{k}$ is the Fourier transform of $k$. (Something similar is true whenever $G$ is compact, but we shall not use this.)

$U$-RS-sequences on $G$ are as defined in 5.4.