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Autor: Edwards, R. E. / Price, J. F.
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§ 10. Concerning the polynomials Q_j .

There is no difficulty in making fairly explicit the construction of t.p.s Q_j of the type employed in 7.6.

For $p > 0$, $t \geq 0$ define

$$h_p(t) = \begin{cases} 1 & \text{if } t \leq p, \\ 2 \left(1 - \frac{t}{2p}\right) & \text{if } p \leq t \leq 2p, \\ 0 & \text{if } t \geq 2p. \end{cases} \quad (10.1)$$

For all complex z define

$$f_p(z) = \begin{cases} 0 & \text{if } z = 0, \\ |z|^{-1} \bar{z} h_p(|z|) & \text{if } z \neq 0. \end{cases} \quad (10.2)$$

Write

$$\left. \begin{aligned} E_n(z) &= \pi^{-1} n \exp(-n|z|^2), \\ P_{n,k}(z) &= \pi^{-1} n \sum_{j=0}^k \frac{(-1)^j}{j!} (n|z|^2)^j \end{aligned} \right\} \quad (10.3)$$

Let μ denote Lebesgue measure on C (identified with R^2 in the canonical fashion).

It is then routine to verify that

$$\left. \begin{aligned} \|E_n * f_p\|_\infty &\leq \|f_p\|_\infty = 1, \\ \lim_{n \rightarrow \infty} E_n * f_p &= f_p \end{aligned} \right\} \quad (10.4)$$

uniformly on any compact set omitting 0. From this it follows that to every $p > 0$ and every positive integer v correspond positive integers $\bar{n}(p, v)$, $\bar{k}(p, v)$ such that

$$\left. \begin{aligned} \left| |z|^{-1} \bar{z} - f_p * P_{\bar{n}, \bar{k}}(z) \right| &\leq \frac{1}{v} \text{ for } \frac{1}{v} \leq |z| \leq p, \\ \left| f_p * P_{\bar{n}, \bar{k}}(z) \right| &\leq 1 + \frac{1}{v} \text{ for } |z| \leq p. \end{aligned} \right\} \quad (10.5)$$

Now

$$f_p * P_{\bar{n}, \bar{k}}(z) = q_{p,v}(z, \bar{z}), \quad (10.6)$$

where

$$\begin{aligned}
 q_{p,v}(X, Y) &= \pi^{-1} \bar{n}(p, v) \sum_{j=0}^{\bar{k}(p,v)} \frac{(-\bar{n}(p, v))^j}{j!} \sum_{l=0}^j \sum_{m=0}^j \binom{j}{l} \binom{j}{m} X^l Y^m \\
 &\quad (-1)^{l+m} \int \zeta^{j-l} \bar{\zeta}^{j-m} f_p(\zeta) d\mu(\zeta) \\
 &= \sum_{l,m=0}^{\bar{k}(p,v)} C_{p,v}(l, m) X^l Y^m.
 \end{aligned} \tag{10.7}$$

It is easily verifiable that the $C_{p,v}(l, m)$ are real-valued.

If θ is a bounded measurable function on G and

$$Q_{p,v}^\circ = q_{p,v}(\theta, \bar{\theta}), \quad p \geq \|\theta\|_\infty, \tag{10.8}$$

we have from (10.5)

$$\left. \begin{aligned}
 \left| |\theta|^{-1} \bar{\theta} - Q_{p,v}^\circ \right| &\leq \frac{1}{v} \text{ whenever } |\theta| \geq \frac{1}{v}, \\
 |Q_{p,v}^\circ| &\leq 1 + \frac{1}{v} \text{ everywhere on } G.
 \end{aligned} \right\} \tag{10.9}$$

If θ is a t.p., then $Q_{p,v}^\circ$ is a t.p. and

$$\text{sp}(Q_{p,v}^\circ) \subseteq [\text{sp}(\theta)]. \tag{10.10}$$

From (10.9) we obtain

$$\left| |\theta| - \theta Q_{p,v}^\circ \right| \leq \begin{cases} v^{-1} |\theta| & \text{whenever } |\theta| \geq \frac{1}{v}, \\ \left(2 + \frac{1}{v}\right) |\theta| & \text{everywhere,} \end{cases}$$

whence it follows that, if $\theta \neq 0$,

$$\begin{aligned}
 \left| \int_G \theta Q_{p,v}^\circ d\lambda_G \right| &\geq (1 - v^{-1}) \|\theta\|_1 - v^{-1} (2 + v^{-1}) \\
 &\geq (1 - 2v^{-\frac{1}{2}}) \|\theta\|_1
 \end{aligned} \tag{10.11}$$

provided $v \geq 9 \|\theta\|_1^{-2}$.

Taking $\theta = D_{\Delta_j}$ and $p_j \geq \|D_{\Delta_j}\|$, the trigonometric polynomials

$$Q_j' = \left(1 + \frac{1}{v}\right)^{-1} Q_{p_j,v}^\circ = \left(1 + \frac{1}{v}\right)^{-1} q_{p_j,v}(D_{\Delta_j}, \bar{D}_{\Delta_j}) \tag{10.12}$$

are then seen from (10.9), (10.10) and (10.11) to satisfy

$$\left. \begin{aligned} \| Q'_j \| &\leq 1, \\ \text{sp } (Q'_j) &\subseteq [A_j], \\ \left| \int v D_{\Delta_j} Q'_j d\lambda_G \right| &\geq (1 - 3v^{-\frac{1}{2}}) \| D_{\Delta_j} \|_1 \end{aligned} \right\} \quad (10.13)$$

provided v is chosen $\geq 9 \| D_{\Delta_j} \|_1^{-1}$. In view of (7.6), we may choose the integer $v \geq \max_j (36, 9 \| D_{\Delta_j} \|_1^{-1})$. Then (10.13) shows that there are unimodular complex numbers ξ_j such that the $Q_j = \xi_j Q'_j$ satisfy (7.7).

APPENDIX

Rudin-Shapiro sequences

A.1 NOTATIONS AND DEFINITIONS. As hitherto, all topological groups G are assumed to be Hausdorff; and, for any locally compact group G , λ_G will denote a selected left Haar measure, with respect to which the Lebesgue spaces $L^p(G)$ are to be formed. $C_c(G)$ denotes the set of complex-valued continuous functions on G having compact supports.

If X and Y are topological groups, $\text{Hom}(X, Y)$ denotes the set of continuous homomorphisms of X into Y .

Suppose henceforth G to be locally compact. As in 5.1, if $k \in C_c(G)$, T_k will denote the convolution operator

$$f \mapsto f * k$$

with domain $C_c(G)$ and range in $C_c(G)$; and $\| k \|_{p,q}$ will denote the (p, q) -norm of this operator, i.e., the smallest real number $m \geq 0$ such that

$$\| f * k \|_q \leq m \| f \|_p \quad (f \in C_c(G)).$$

It is well-known that, if G is Abelian, $\| k \|_{2,2}$ is equal to

$$\| \hat{k} \|_\infty = \sup_{\gamma \in \Gamma} | \hat{k}(\gamma) |,$$

where Γ is the character group of G and \hat{k} is the Fourier transform of k . (Something similar is true whenever G is compact, but we shall not use this.)

U -RS-sequences on G are as defined in 5.4.