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$$-280 - u_n = n^{-2} Q_{j_n}$$

satisfy the conditions

$$\sup (u_n) \subseteq \Gamma_0, \sum_{n=1}^{\infty} || u_n || < \infty
 S_{\Delta_{j_n}} u_n (0) \text{ is real and } > n.$$
(7.9)

At this point the construction in § 2 will yield integers $0 < n_1 < n_2 < ...$ and specifiable sequences $(\gamma_p)_{p \in N}$ of positive numbers such that each function of the form

$$f = \sum_{p=1}^{\infty} \gamma_p \, u_{n_p}$$

is continuous and satisfies

$$sp(f) \subseteq \Gamma_0, \lim_{p \to \infty} \operatorname{Re} S_{A_{j_n}} f(0) = \infty.$$
 (7.10)

A fortiori, f satisfies (7.3).

We add here that, if the Δ_j are symmetric, the D_{Δ_j} are real-valued, and we may work throughout with real-valued functions, replacing Re $S_{\Delta_j} f$ by $S_{\Delta_j} f$ everywhere.

§8. Discussion of case (i): G not 0-dimensional

8.1 In this case $\Phi \neq \Gamma$, and we begin by considering a finite subset of Γ of the form \cdot

$$\Delta = \Omega + \Lambda, \tag{8.1}$$

where Ω and Λ are finite subsets of Γ such that $\pi \mid \Omega$ is 1-1 and $\emptyset \neq \Lambda \subseteq \Phi$. We aim to show that (for a suitable absolute constant k > 0)

$$|| D_{\Delta} ||_{1} \ge k \left(\frac{\log N}{\log \log N} \right)^{\frac{1}{4}}, \qquad (8.2)$$

provided $N = |\Omega|$ (the cardinal number of Ω) is sufficiently large.

8.2 PROOF OF (8.2). Introduce H as the annihilator in G of Φ and identify in the usual way the dual of H with Γ/Φ . Likewise identify the dual of K = G/H with Φ ([7], (24.11)).

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We then have

$$| D_{\Delta} | |_{1} = \int_{G} | \sum_{\gamma \in \Delta} \gamma | d\lambda_{G}$$

= $\int_{G/H} d\lambda_{G/H}(\bar{x}) \int_{H} | \sum_{\theta \in \Omega} \sum_{\phi \in \Lambda} \theta (x+y) \phi (x+y) | d\lambda_{H}(y),$

the inner integral being viewed as a function of $\bar{x} = x + H$ Thus, writing $\bar{\theta}$ for $\pi(\theta)$ and noting that $\phi(y) = 1$ for $\phi \in \Lambda \subseteq \Phi$ and $y \in H$, we obtain

$$\| D_{\Delta} \|_{1} = \int_{G/H} d\lambda_{G/H}(\bar{x}) \int_{H} | \sum_{\theta \in \Omega} \alpha(\theta, x) \bar{\theta}(y) | d\lambda_{H}(y), \qquad (8.3)$$

where

$$\alpha(\theta, x) = \theta(x) \sum_{\phi \in \Lambda} \phi(x).$$

Now, since the dual of H (namely Γ/Φ) is torsion-free ([7], (A.4)), Theorem A of [8] shows that (for a suitable absolute constant k > 0) we have

$$\begin{split} \int_{H} \Big| \sum_{\theta \in \Omega} \alpha(\theta, x) \overline{\theta}(y) \Big| d\lambda_{H}(y) &\geq k \left(\frac{\log N}{\log \log N} \right)^{\frac{1}{4}} \min_{\theta \in \Omega} \Big| \alpha(\theta, x) \Big| \\ &= k \left(\frac{\log N}{\log \log N} \right)^{\frac{1}{4}} \Big| \sum_{\phi \in \Lambda} \phi(\overline{x}) \Big|, \end{split}$$
(8.4)

since $|\theta(x)| = 1$ and $\phi(x)$ depends only \bar{x} . By (8.3) and (8.4),

$$|| D_{\mathcal{A}} ||_{1} \geq k \left(\frac{\log N}{\log \log N} \right)^{\frac{1}{4}} \int_{G/H} | \sum_{\phi \in \mathcal{A}} \phi(\bar{x}) | d\lambda_{G/H}(\bar{x}).$$
(8.5)

Since $\Lambda \neq \emptyset$, the remaining integral is not less than the maximum modulus of the Fourier transform of the function $\bar{x} \mapsto \sum_{\phi \in \Lambda} \phi(\bar{x})$, i.e., is not less than unity. Thus, (8.2) follows from (8.5).

8.3 PROOF OF 7.4 (i). The conclusions stated in case (i) of 7.4 are now almost immediate. If $\mathscr{D} = (\varDelta_j)_{j \in \mathbb{N}}$ is a grouping of infinite type covering Γ_0 , $|\pi(\varDelta_j)| \to \infty$ and so, since $\varDelta_j \subseteq \Phi$, $|\pi(\Omega_j)| \to \infty$. Then (8.2) shows that (7.6) is satisfied, and it remains only to refer to 7.6.

8.4 SUPPLEMENTARY REMARKS. The fact that, when G is not 0-dimensional, (7.6) holds for suitable subgroups Γ_0 of Γ and suitable groupings $\mathcal{D} = (\Delta_j)_{j \in \mathbb{N}}$ covering Γ_0 can be derived without appeal to Theorem A

the family $(\gamma_k)_{1 \le k \le m}$ is independent (see [7], (A.10)), define

$$\Gamma_0 = \{ \sum_{k=1}^m n_k \gamma_k : n_k \in Z \text{ for } k = 1, 2, ..., m \},\$$

and make use of the formula

$$\int_{G} F(\gamma_{1}(x), ..., \gamma_{m}(x)) d\gamma_{G}(x) = (2\pi)^{-m} \int_{0}^{2\pi} ... \int_{0}^{2\pi} F(e^{it}, ..., e^{it_{m}}) dt_{1} ... dt_{m}, \qquad (8.6)$$

valid for every $F \in C(T^m)$, where T denotes the circle group. (Recall that $\sum_{k=1}^{m} n_k \gamma_k$ denotes the character $x \to \gamma_1(x)^{n_1} \dots \gamma_m(x)^{n_m}$ of G.) It then appears that (7.6) holds when one takes

$$\Delta_{j} = \{ \sum_{k=1}^{m} n_{k} \gamma_{k} : |n_{k}| \leq r_{j,k} \text{ for } k = 1, 2, ..., m \},\$$

where the $r_{j,k}$ are positive integers satisfying $r_{j,k} \leq r_{j,k+1}$ and $\lim_{j\to\infty} r_{j,k} = \infty$. Moreover, when m = 1, the Cohen-Davenport result (essentially Theorem A of [8] for the case G = T) shows that (7.6) holds for every grouping \mathcal{D} covering Γ_0 .

The verification of (8.6) is simple. First note that, if G and G' are compact groups, and if ϕ is a continuous homomorphism of G into G', then

$$\int_{G} (F \circ \phi) d\lambda_{G} = \int F d\lambda_{\phi(G)}$$
(8.7)

for every $F \in C(G')$. (This is a consequence of the fact that $F \mid \rightarrow \int_{G} (F \circ \phi) d\lambda_{G}$ is invariant under translation by elements of $\phi(G)$, combined with the uniqueness of the normalised Haar measure on a compact group.) Taking $G' = T^{m}$ and $\phi: x \mid \rightarrow (\gamma_{1}(x), ..., \gamma_{m}(x))$, the stated conditions on the γ_{k} are just adequate to ensure that the annihilator in Z^{m} (identified in the canonical fashion with the dual of T^{m}) of $\phi(G)$ is $\{(0, ..., 0)\}$ and so ([7], (24.10)) that $\phi(G) = T^{m}$. Accordingly, (8.6) appears as a special case of (8.7).

It is perhaps worth indicating that special cases of (8.7) can be exploited in other ways. For example, suppose more generally that κ is an arbitrary nonvoid set and that $(\gamma_k)_{k\in\kappa}$ is a finite or infinite independent family of elements of $\Gamma \setminus \Phi$. Denote by Γ_0 the subgroup of Γ generated by $\{\gamma_k : k \in \kappa\}$. Taking $G' = T^{\kappa}$ and $\phi : x \to (\gamma_k(x))_{k\in\kappa}$, one may use (8.7) in a similar fashion to show that there is an isometric isomorphism $F \leftrightarrow F \circ \phi = f$ between $L^p(T^{\kappa})$ (or $C(T^{\kappa})$) and the subspace of $L^p(G)$ (or C(G)) formed of those $f \in L^p(G)$ or C(G) such that sp $(f) \subseteq \Gamma_0$. Moreover, if one identifies in the canonical fashion the dual of T^{κ} with the weak — 283 —

direct product Z^{κ}^* , the said isomorphism is such that $\hat{F} = \hat{f} \circ \phi'$, where ϕ' is the isomorphism of Z^{κ}^* onto Γ_0 defined by $(n_k) \to \sum_{k \in \kappa} n_k \gamma_k$.

One consequence of this may be expressed roughly as follows: If the compact Abelian group G is such that $\Gamma \setminus \Phi$ contains an independent family of (finite or infinite) cardinality m, then Fourier series on G behave, in respect of convergence or summability, no better than do Fourier series on T^m .

Another consequence is that, if Δ is a subset of Γ_0 , then Δ is a Sidon (or $\Lambda(p)$) subset of Γ if and only if ${\phi'}^{-1}(\Delta)$ is a Sidon (or $\Lambda(p)$) subset of Z^{κ}^* .

8.5 FURTHER RESULTS. Theorem A of [8] implies something stronger than (8.2), namely: if ω is any complex-valued function on Γ such that

$$\omega (\gamma + \phi) = \omega (\gamma) \quad (\gamma \in \Gamma, \phi \in \Phi), \tag{8.8}$$

so that ω can be regarded as a function on Γ/Φ , and if we write

$$D^{\omega}_{\Delta} = \sum_{\gamma \in \Delta} \omega(\gamma) \,\bar{\gamma}, \, S^{\omega}_{\Delta} f = \sum_{\gamma \in \Delta} \omega(\gamma) \,\hat{f}(\gamma), \quad (8.9)$$

then, for $\Delta = \Omega + \Lambda$ as in (8.1), we have

$$\left\| D_{\Delta}^{\omega} \right\|_{1} \ge k \left(\frac{\log N}{\log \log N} \right)^{\frac{1}{4}} \min_{\gamma \in \Omega} \left| \omega \left(\gamma \right) \right|$$
(8.10)

provided $N = |\Omega|$ is sufficiently large.

So, if we can arrange for $\Omega = \Omega_j$ to vary in such a way that the righthand side of (8.10) tends to infinity with *j*, the substance of 7.6 will lead to a continuous *f* satisfying sp $(f) \subseteq \Gamma_0$ and

$$\lim_{j \to \infty} \operatorname{Re} S^{\omega}_{\Delta_j} f(0) = \infty.$$
(8.11)

Taking the most familiar case, in which G = T, $\Gamma = Z$ and $\Phi = \{0\}$, and supposing $\Delta = \Omega$ to range over a sequence (Δ_j) of finite subsets of Z such that, if $N_j = |\Delta_j|$,

$$\lim_{j} \left(\frac{\log N_j}{\log \log N_j} \right)^{\frac{1}{4}} \min_{n \in \mathcal{A}_j} |\omega(n)| = \infty,$$

the construction will lead to a continuous f on T such that

$$\overline{\lim_{j}} \operatorname{Re} S^{\omega}_{\Delta_{j}} f(0) = \infty.$$

In particular, taking $\Delta_j = \{n \in Z : 2^j \leq n < 2^{j+1}\}$ it can be arranged that

$$\sum_{n\in\mathbb{Z}}\frac{\pm \hat{f}(n)}{(\log\left(2+\left|n\right|\right))^{\alpha}}$$

diverges for any preassigned distribution of signs \pm and any preassigned $\alpha < \frac{1}{4}$.

Of course, much stronger results are derivable by using random (and unspecifiable!) changes of sign, but there seems little hope of making this even remotely constructive.

§9. Discussion of case (ii): G 0-dimensional

9.1 In this case there is ([7], (7.7)) a base of neighbourhoods of zero in G formed of compact open subgroups W. For each such W the annihilator $\Delta = W^{\circ}$ in Γ of W is a finite subgroup of Γ . Define

$$k_W = \lambda_G(W)^{-1} \times \text{characteristic function of } W.$$
 (9.1)

Then k_W is continuous, $k_W \ge 0$, $\int_G k_W d\lambda_G = 1$. The transform k_W of k_W is plainly equal to unity on Δ . On the other hand, since W is a subgroup, we have for $a \in W$ and $\gamma \in \Gamma$

$$\hat{k}_{W}(\gamma) = \int_{G} k_{W}(x) \overline{\gamma(x)} d\lambda_{G}(x) = \int_{G} k_{W}(x+a) \overline{\gamma(x)} d\lambda_{G}(x)$$
$$= \int_{G} k_{W}(y) \overline{\gamma(y-a)} d\lambda_{G}(y)$$
$$= \gamma(a) \hat{k}_{W}(\gamma),$$

which shows that $\hat{k}_{W}(\gamma) = 0$ if $\gamma \in \Gamma \setminus \Delta$. Thus \hat{k}_{W} is the characteristic function of Δ , and so

$$k_{W} = D_{W^{\circ}}. \tag{9.2}$$

By (9.1) and (9.2), a routine argument shows that, if $1 \leq p < \infty$ and $f \in L^p(G)$, then

$$f = \lim_{W} S_{W^{\circ}} f \tag{9.3}$$

in $L^{p}(G)$; and that (9.3) holds uniformly for any continuous f.

9.2 PROOF OF 7.4 (ii). If Γ_0 is any countably infinite subgroup of Γ we can choose a sequence W_j of compact open subgroups of G such that