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<b>Autor:</b>	Edwards, R. E. / Price, J. F.
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$$u_n = n^{-2} Q_{j_n}$$

satisfy the conditions

$$\left. \begin{array}{l} \text{sp}(u_n) \subseteq \Gamma_0, \sum_{n=1}^{\infty} \|u_n\| < \infty \\ S_{A_{j_n}} u_n(0) \text{ is real and } > n. \end{array} \right\} \quad (7.9)$$

At this point the construction in § 2 will yield integers  $0 < n_1 < n_2 < \dots$  and specifiable sequences  $(\gamma_p)_{p \in N}$  of positive numbers such that each function of the form

$$f = \sum_{p=1}^{\infty} \gamma_p u_{n_p}$$

is continuous and satisfies

$$\text{sp}(f) \subseteq \Gamma_0, \lim_{p \rightarrow \infty} \text{Re } S_{A_{j_{n_p}}} f(0) = \infty. \quad (7.10)$$

A fortiori,  $f$  satisfies (7.3).

We add here that, if the  $A_j$  are symmetric, the  $D_{A_j}$  are real-valued, and we may work throughout with real-valued functions, replacing  $\text{Re } S_{A_j} f$  by  $S_{A_j} f$  everywhere.

### § 8. Discussion of case (i) : $G$ not 0-dimensional

8.1 In this case  $\Phi \neq \Gamma$ , and we begin by considering a finite subset of  $\Gamma$  of the form

$$A = \Omega + \Lambda, \quad (8.1)$$

where  $\Omega$  and  $\Lambda$  are finite subsets of  $\Gamma$  such that  $\pi \mid \Omega$  is 1-1 and  $\emptyset \neq \Lambda \subseteq \Phi$ . We aim to show that (for a suitable absolute constant  $k > 0$ )

$$\|D_A\|_1 \geq k \left( \frac{\log N}{\log \log N} \right)^{\frac{1}{4}}, \quad (8.2)$$

provided  $N = |\Omega|$  (the cardinal number of  $\Omega$ ) is sufficiently large.

8.2 PROOF OF (8.2). Introduce  $H$  as the annihilator in  $G$  of  $\Phi$  and identify in the usual way the dual of  $H$  with  $\Gamma/\Phi$ . Likewise identify the dual of  $K = G/H$  with  $\Phi$  ([7], (24.11)).

We then have

$$\begin{aligned}\|D_A\|_1 &= \int_G \left| \sum_{\gamma \in A} \gamma \right| d\lambda_G \\ &= \int_{G/H} d\lambda_{G/H}(\bar{x}) \int_H \left| \sum_{\theta \in \Omega} \sum_{\phi \in A} \theta(x+y) \phi(x+y) \right| d\lambda_H(y),\end{aligned}$$

the inner integral being viewed as a function of  $\bar{x} = x+H$ . Thus, writing  $\bar{\theta}$  for  $\pi(\theta)$  and noting that  $\phi(y) = 1$  for  $\phi \in A \subseteq \Phi$  and  $y \in H$ , we obtain

$$\|D_A\|_1 = \int_{G/H} d\lambda_{G/H}(\bar{x}) \int_H \left| \sum_{\theta \in \Omega} \alpha(\theta, x) \bar{\theta}(y) \right| d\lambda_H(y), \quad (8.3)$$

where

$$\alpha(\theta, x) = \theta(x) \sum_{\phi \in A} \phi(x).$$

Now, since the dual of  $H$  (namely  $\Gamma/\Phi$ ) is torsion-free ([7], (A.4)), Theorem A of [8] shows that (for a suitable absolute constant  $k > 0$ ) we have

$$\begin{aligned}\int_H \left| \sum_{\theta \in \Omega} \alpha(\theta, x) \bar{\theta}(y) \right| d\lambda_H(y) &\geq k \left( \frac{\log N}{\log \log N} \right)^{\frac{1}{4}} \min_{\theta \in \Omega} |\alpha(\theta, x)| \\ &= k \left( \frac{\log N}{\log \log N} \right)^{\frac{1}{4}} \left| \sum_{\phi \in A} \phi(\bar{x}) \right|,\end{aligned} \quad (8.4)$$

since  $|\theta(x)| = 1$  and  $\phi(x)$  depends only  $\bar{x}$ . By (8.3) and (8.4),

$$\|D_A\|_1 \geq k \left( \frac{\log N}{\log \log N} \right)^{\frac{1}{4}} \int_{G/H} \left| \sum_{\phi \in A} \phi(\bar{x}) \right| d\lambda_{G/H}(\bar{x}). \quad (8.5)$$

Since  $A \neq \emptyset$ , the remaining integral is not less than the maximum modulus of the Fourier transform of the function  $\bar{x} \mapsto \sum_{\phi \in A} \phi(\bar{x})$ , i.e., is not less than unity. Thus, (8.2) follows from (8.5).

**8.3 PROOF OF 7.4 (i).** The conclusions stated in case (i) of 7.4 are now almost immediate. If  $\mathcal{D} = (A_j)_{j \in N}$  is a grouping of infinite type covering  $\Gamma_0$ ,  $|\pi(A_j)| \rightarrow \infty$  and so, since  $A_j \subseteq \Phi$ ,  $|\pi(\Omega_j)| \rightarrow \infty$ . Then (8.2) shows that (7.6) is satisfied, and it remains only to refer to 7.6.

**8.4 SUPPLEMENTARY REMARKS.** The fact that, when  $G$  is not 0-dimensional, (7.6) holds for suitable subgroups  $\Gamma_0$  of  $\Gamma$  and suitable groupings  $\mathcal{D} = (A_j)_{j \in N}$  covering  $\Gamma_0$  can be derived without appeal to Theorem A

of [8]. To do this, it suffices to take  $\gamma_k \in \Gamma \setminus \Phi$  ( $k = 1, 2, \dots, m$ ) such that the family  $(\gamma_k)_{1 \leq k \leq m}$  is independent (see [7], (A.10)), define

$$\Gamma_0 = \left\{ \sum_{k=1}^m n_k \gamma_k : n_k \in \mathbb{Z} \text{ for } k = 1, 2, \dots, m \right\},$$

and make use of the formula

$$\begin{aligned} \int_G F(\gamma_1(x), \dots, \gamma_m(x)) d\gamma_G(x) \\ = (2\pi)^{-m} \int_0^{2\pi} \dots \int_0^{2\pi} F(e^{it_1}, \dots, e^{it_m}) dt_1 \dots dt_m, \end{aligned} \quad (8.6)$$

valid for every  $F \in C(T^m)$ , where  $T$  denotes the circle group. (Recall that  $\sum_{k=1}^m n_k \gamma_k$  denotes the character  $x \mapsto \gamma_1(x)^{n_1} \dots \gamma_m(x)^{n_m}$  of  $G$ .) It then appears that (7.6) holds when one takes

$$\Delta_j = \left\{ \sum_{k=1}^m n_k \gamma_k : |n_k| \leq r_{j,k} \text{ for } k = 1, 2, \dots, m \right\},$$

where the  $r_{j,k}$  are positive integers satisfying  $r_{j,k} \leq r_{j,k+1}$  and  $\lim_{j \rightarrow \infty} r_{j,k} = \infty$ . Moreover, when  $m = 1$ , the Cohen-Davenport result (essentially Theorem A of [8] for the case  $G = T$ ) shows that (7.6) holds for every grouping  $\mathcal{D}$  covering  $\Gamma_0$ .

The verification of (8.6) is simple. First note that, if  $G$  and  $G'$  are compact groups, and if  $\phi$  is a continuous homomorphism of  $G$  into  $G'$ , then

$$\int_G (F \circ \phi) d\lambda_G = \int F d\lambda_{\phi(G)} \quad (8.7)$$

for every  $F \in C(G')$ . (This is a consequence of the fact that  $F \mapsto \int_G (F \circ \phi) d\lambda_G$  is invariant under translation by elements of  $\phi(G)$ , combined with the uniqueness of the normalised Haar measure on a compact group.) Taking  $G' = T^m$  and  $\phi : x \mapsto (\gamma_1(x), \dots, \gamma_m(x))$ , the stated conditions on the  $\gamma_k$  are just adequate to ensure that the annihilator in  $\mathbb{Z}^m$  (identified in the canonical fashion with the dual of  $T^m$ ) of  $\phi(G)$  is  $\{(0, \dots, 0)\}$  and so ([7], (24.10)) that  $\phi(G) = T^m$ . Accordingly, (8.6) appears as a special case of (8.7).

It is perhaps worth indicating that special cases of (8.7) can be exploited in other ways. For example, suppose more generally that  $\kappa$  is an arbitrary nonvoid set and that  $(\gamma_k)_{k \in \kappa}$  is a finite or infinite independent family of elements of  $\Gamma \setminus \Phi$ . Denote by  $\Gamma_0$  the subgroup of  $\Gamma$  generated by  $\{\gamma_k : k \in \kappa\}$ . Taking  $G' = T^\kappa$  and  $\phi : x \mapsto (\gamma_k(x))_{k \in \kappa}$ , one may use (8.7) in a similar fashion to show that there is an isometric isomorphism  $F \leftrightarrow F \circ \phi = f$  between  $L^p(T^\kappa)$  (or  $C(T^\kappa)$ ) and the subspace of  $L^p(G)$  (or  $C(G)$ ) formed of those  $f \in L^p(G)$  or  $C(G)$  such that  $\text{sp}(f) \subseteq \Gamma_0$ . Moreover, if one identifies in the canonical fashion the dual of  $T^\kappa$  with the weak

direct product  $Z^\kappa^*$ , the said isomorphism is such that  $\hat{F} = \hat{f} \circ \phi'$ , where  $\phi'$  is the isomorphism of  $Z^\kappa^*$  onto  $\Gamma_0$  defined by  $(n_k) \rightarrow \sum_{k \in \kappa} n_k \gamma_k$ .

One consequence of this may be expressed roughly as follows: If the compact Abelian group  $G$  is such that  $\Gamma \setminus \Phi$  contains an independent family of (finite or infinite) cardinality  $m$ , then Fourier series on  $G$  behave, in respect of convergence or summability, no better than do Fourier series on  $T^m$ .

Another consequence is that, if  $\Delta$  is a subset of  $\Gamma_0$ , then  $\Delta$  is a Sidon (or  $\Lambda(p)$ ) subset of  $\Gamma$  if and only if  $\phi'^{-1}(\Delta)$  is a Sidon (or  $\Lambda(p)$ ) subset of  $Z^\kappa^*$ .

**8.5 FURTHER RESULTS.** Theorem A of [8] implies something stronger than (8.2), namely: if  $\omega$  is any complex-valued function on  $\Gamma$  such that

$$\omega(\gamma + \phi) = \omega(\gamma) \quad (\gamma \in \Gamma, \phi \in \Phi), \quad (8.8)$$

so that  $\omega$  can be regarded as a function on  $\Gamma/\Phi$ , and if we write

$$D_\Delta^\omega = \sum_{\gamma \in \Delta} \omega(\gamma) \bar{\gamma}, \quad S_\Delta^\omega f = \sum_{\gamma \in \Delta} \omega(\gamma) \hat{f}(\gamma), \quad (8.9)$$

then, for  $\Delta = \Omega + \Lambda$  as in (8.1), we have

$$\|D_\Delta^\omega\|_1 \geq k \left( \frac{\log N}{\log \log N} \right)^{\frac{1}{4}} \min_{\gamma \in \Omega} |\omega(\gamma)| \quad (8.10)$$

provided  $N = |\Omega|$  is sufficiently large.

So, if we can arrange for  $\Omega = \Omega_j$  to vary in such a way that the right-hand side of (8.10) tends to infinity with  $j$ , the substance of 7.6 will lead to a continuous  $f$  satisfying  $\text{sp}(f) \subseteq \Gamma_0$  and

$$\overline{\lim}_{j \rightarrow \infty} \text{Re } S_{\Delta_j}^\omega f(0) = \infty. \quad (8.11)$$

Taking the most familiar case, in which  $G = T$ ,  $\Gamma = Z$  and  $\Phi = \{0\}$ , and supposing  $\Delta = \Omega$  to range over a sequence  $(\Delta_j)$  of finite subsets of  $Z$  such that, if  $N_j = |\Delta_j|$ ,

$$\lim_j \left( \frac{\log N_j}{\log \log N_j} \right)^{\frac{1}{4}} \min_{n \in \Delta_j} |\omega(n)| = \infty,$$

the construction will lead to a continuous  $f$  on  $T$  such that

$$\overline{\lim}_j \text{Re } S_{\Delta_j}^\omega f(0) = \infty.$$

In particular, taking  $\Delta_j = \{n \in \mathbb{Z} : 2^j \leq n < 2^{j+1}\}$  it can be arranged that

$$\sum_{n \in \mathbb{Z}} \frac{\pm \hat{f}(n)}{(\log(2 + |n|))^\alpha}$$

diverges for any preassigned distribution of signs  $\pm$  and any preassigned  $\alpha < \frac{1}{4}$ .

Of course, much stronger results are derivable by using random (and unspecifiable!) changes of sign, but there seems little hope of making this even remotely constructive.

### § 9. Discussion of case (ii) : $G$ 0-dimensional

9.1 In this case there is ([7], (7.7)) a base of neighbourhoods of zero in  $G$  formed of compact open subgroups  $W$ . For each such  $W$  the annihilator  $\Delta = W^\circ$  in  $\Gamma$  of  $W$  is a finite subgroup of  $\Gamma$ . Define

$$k_W = \lambda_G(W)^{-1} \times \text{characteristic function of } W. \quad (9.1)$$

Then  $k_W$  is continuous,  $k_W \geq 0$ ,  $\int_G k_W d\lambda_G = 1$ . The transform  $\hat{k}_W$  of  $k_W$  is plainly equal to unity on  $\Delta$ . On the other hand, since  $W$  is a subgroup, we have for  $a \in W$  and  $\gamma \in \Gamma$

$$\begin{aligned} \hat{k}_W(\gamma) &= \int_G k_W(x) \overline{\gamma(x)} d\lambda_G(x) = \int_G k_W(x+a) \overline{\gamma(x)} d\lambda_G(x) \\ &= \int_G k_W(y) \overline{\gamma(y-a)} d\lambda_G(y) \\ &= \gamma(a) \hat{k}_W(\gamma), \end{aligned}$$

which shows that  $\hat{k}_W(\gamma) = 0$  if  $\gamma \in \Gamma \setminus \Delta$ . Thus  $\hat{k}_W$  is the characteristic function of  $\Delta$ , and so

$$k_W = D_{W^\circ}. \quad (9.2)$$

By (9.1) and (9.2), a routine argument shows that, if  $1 \leq p < \infty$  and  $f \in L^p(G)$ , then

$$f = \lim_w S_{W^\circ} f \quad (9.3)$$

in  $L^p(G)$ ; and that (9.3) holds uniformly for any continuous  $f$ .

9.2 PROOF OF 7.4 (ii). If  $\Gamma_0$  is any countably infinite subgroup of  $\Gamma$  we can choose a sequence  $W_j$  of compact open subgroups of  $G$  such that