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PRINCIPLES, WITH APPLICATIONS TO HARMONIC ANALYSIS

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satisfying $1 \le p < 2 < q \le \infty$, the series (6.6) converges normally in $L_p^q(G)$ to T. Next, T is the limit in E of

$$S_r = \sum_{n=1}^r \omega_n \, T_{K_n}$$

as $r \to \infty$ and, since it is plain that supp $S_r \subseteq \Omega$ for every r, (ii) is easily derived. Finally, if \hat{T} were a measure μ , it would necessarily be the case that supp $\mu \subseteq \overline{\Omega}$ and so, for every $n \in N$, one would have by (6.1) and (6.4)

$$f_n(T) = |u_n * Tv_n(0)| = |\int_{\Gamma} \hat{u}_n \hat{v}_n d\mu|$$

$$\leq |\mu|(\overline{\Omega}),$$

which is finite since Ω is relatively compact. However, this plainly would entail $f^*(T) < \infty$, in conflict with (6.8), so that T cannot be a measure and (iii) is verified. This completes the proof.

6.4 Remark. Theorem 6.3 was proved by Hörmander ([14], Theorem 1.9) for $G = R^n$ and any given pair (p, q) satisfying $1 \le p < 2 < q \le \infty$, this result being extended to a general noncompact LCA G by Gaudry [5]. The argument given by Hörmander (loc. cit. Theorem 1.6 and the remark immediately following) for the case $G = R^n$ can also be extended to a general LCA G and shows that, if either $q \le 2$ or $p \ge 2$, then every $T \in L^q_p(G)$ is such that \hat{T} is a measure [and indeed a measure of the form $\psi \lambda_{\Gamma}$, where $\psi \in L^2_{loc}(\Gamma)$ if $q \le 2$ and $\psi \in L^p_{loc}(\Gamma)$ if $p \ge 2$, and so $\psi \in L^2_{loc}(\Gamma)$ in either case]. Thus the hypotheses made in Theorem 6.3 about p and q are necessary for the validity of the conclusion.

PART 3: APPLICATIONS TO FOURIER SERIES

§ 7. Applications to divergence of Fourier series.

7.1 Throughout §§ 7-10, G will denote an infinite Hausdorff compact Abelian group with character group Γ , and λ_G the Haar measure on G, normalised so that $\lambda_G(G) = 1$. For any $f \in L^1(G)$, \hat{f} will denote the Fourier transform of f; for any finite subset Δ of Γ ,

$$S_{\Delta}f = \sum_{\gamma \in \Delta} \hat{f}(\gamma)\gamma \tag{7.1}$$

is the Δ -partial sum of the Fourier series of f; and sp (f) will stand for

the spectrum of f, i.e., for the support supp $\hat{f} = \{ \gamma \in \Gamma : \hat{f}(\gamma) \neq 0 \}$ of \hat{f} . The term "trigonometric polynomial" will frequently be abbreviated to "t.p.". In addition, Φ will denote the largest torsion subgroup of Γ ([7], (A.4)), and π the natural map of Γ onto Γ/Φ . If Δ denotes a subset of Γ , $[\Delta]$ will stand for the subgroup of Γ generated by Δ .

By a (convergence) grouping we shall mean a sequence $\mathcal{D} = (\Delta_j)_{j \in \mathbb{N}} = (\Delta_j)$ of finite subsets Δ_j of Γ such that

$$\Delta_{j} \subseteq \Delta_{j+1} \quad (j \in N);$$

$$\overset{\circ}{\bigcup} \Delta_{j} = \Gamma_{0} \text{ is a subgroup of } \Gamma, \text{ said to be}$$

$$covered \ by \ \mathcal{D};$$
for each $j \in N$, $\Delta_{j} = \Omega_{j} + \Lambda_{j}$, where Λ_{j} is a nonvoid finite subset of Φ and Ω_{j} is a finite subset of Γ such that $\pi \mid \Omega_{j}$ is 1-1.

[The first two conditions are natural enough in the context described in 7.3, but the third is less so and may well be pointless.] The grouping \mathcal{D} is said to be of *infinite type* if and only if $\pi(\Gamma_0)$ is infinite.

- 7.2 Examples. (i) Let Γ_0 be any countable subgroup of Γ such that $\Gamma_0 \cap \Phi = \{0\}$; for example, $\Gamma_0 = \{n\gamma_0 : n \in Z\}$, where $\gamma_0 \in \Gamma \setminus \Phi$. Then a grouping $\mathscr D$ covering Γ_0 results whenever $\Lambda_j = \{0\}$ and $\Delta_j = \Omega_j$ for every $j \in N$, where $(\Omega_j)_{j \in N}$ is any increasing sequence of finite subsets of Γ_0 with union equal to Γ_0 . This grouping is of infinite type if and only if Γ_0 is infinite.
- (ii) If G is connected, and if Γ_0 is any countable subgroup of Γ , then ([10], 2.5.6 (c), 8.1.2 (a) and (b) and 8.1.6) Γ_0 is an ordered group isomorphic to a discrete subgroup of R. Assuming $\Gamma_0 \neq \{0\}$, Γ_0 has a smallest positive element γ_0 and $\Gamma_0 = \{n\gamma_0 : n \in Z\}$. A natural grouping \mathcal{D} covering Γ_0 is that in which $\Lambda_j = \{0\}$ and

$$\Delta_j = \Omega_j = \{ n\gamma_0 : n \in \mathbb{Z}, \mid n \mid \leq j \}$$

for every $j \in N$; this grouping is of infinite type.

7.3 A grouping $\mathscr{D} = (\Delta_j)_{j \in \mathbb{N}}$ will be thought of as specifying one of the many possible ways in which one may interpret the convergence of Fourier series of functions f on G satisfying $sp(f) \subseteq \Gamma_0$, namely, as convergence of the corresponding sequence of partial sums $(S_{\Delta_j} f)_{j \in \mathbb{N}}$.

Indeed, the conditions (7.2) guarantee that $\lim_{j\to\infty} S_{\Delta_j} f = f$ for all sufficiently regular such functions f. However, our concern rests with the possibility of constructing continuous functions f on G satisfying

$$\operatorname{sp}(f) \subseteq \Gamma_0, \overline{\lim_{j \to \infty}} \operatorname{Re} S_{\Delta_j} f(0) = \infty. \tag{7.3}$$

It will appear that the possibilities exhibit a fairly clear dichotomy, depending largely upon whether G is or is not 0-dimensional.

In the first place, it will emerge in 7.6 that the construction principle of § 2, applied to the Banach space E = C(G) of continuous complex valued functions on G [with norm $||\cdot||$ equal to the maximum modulus] and to sequences of gauges of the type

$$f \mid \to \operatorname{Re} S_{\Delta} f(0) = \operatorname{Re} \int_{G} D_{\Delta} f d\lambda_{G},$$
 (7.4)

where D_A stands for the "Dirichlet function"

$$D_{\Delta} = \sum_{\gamma \in \Delta} \bar{\gamma},\tag{7.5}$$

shows that the problem hinges on the existence of groupings \mathcal{D} for which

$$\rho_j = \| D_{\Delta_j} \|_1 = \int_G | D_{\Delta_j} | d\lambda_G \to \infty.$$
 (7.6)

Accordingly, and in view of the fact ([7], (24.26)) that G is 0-dimensional if and only if Γ coincides with Φ , it emerges that the dichotomy referred to may be expressed in the following way.

7.4 Two cases arise, namely:

- (i) G is not 0-dimensional (i.e., $\Phi \neq \Gamma$). Then (see Example 7.2 (i)) there exist groupings $\mathscr{D} = (\Delta_j)$ of infinite type; and, for any such grouping, one can construct (fairly explicitly, as described in 7.6) continuous functions f on G satisfying (7.3). In particular [cf. Example 7.2 (i)], if Γ_0 is any countably infinite subgroup of Γ satisfying $\Gamma_0 \cap \Phi = \{0\}$, and if $(\Delta_j)_{j \in N}$ is any increasing sequence of finite subsets of Γ_0 with union Γ_0 , we can construct a continuous f on G satisfying (7.3).
- (ii) G is 0-dimensional (i.e., $\Phi = \Gamma$). Then there exists no grouping of infinite type. However, given any countable subgroup Γ_0 of Γ , there are groupings $\mathcal{D} = (\Delta_j)$ covering Γ_0 , in which $\Omega_j = \{0\}$ and $\Delta_j = \Lambda_j$ is a finite subgroup of Γ_0 , and for which

$$f = \lim_{i \to \infty} S_{A_j} f$$

uniformly on G for every continuous f satisfying sp $(f) \subseteq \Gamma_0$.

Case (i) will be dealt with in § 8, case (ii) in § 9. The groupings described in case (ii) prove to be exceptional in various ways; see 9.3.

7.5 REMARK. Perhaps it should be stressed here that, if Γ_0 is any infinite subgroup of Γ , there is no obstacle to constructing continuous functions f such that $\operatorname{sp}(f) \subseteq \Gamma_0$ and finite subsets $\Delta_j \subseteq \Delta_{j+1}$ of Γ_0 for which

$$\lim_{i} S_{A_{j}} f(0) = \infty.$$

[One has in fact only to construct a continuous f such that $\operatorname{sp}(f) \subseteq \Gamma_0$ and $\sum_{\gamma \in \Gamma} |\hat{f}(\gamma)| = \infty$; it is then trivial that there exist finite subsets Δ of Γ_0 for which $|S_{\Delta}f(0)|$ is arbitrarily large, so that we can choose a sequence (Δ_j) for which $\Delta_j \subseteq \Delta_{j+1}$ and $|S_{\Delta_j}f(0)| \to \infty$ with j.] However, the sets Δ_j obtained this way will not [and, in view of 7.4 (ii), cannot] in general be such that $\bigcup_{j=1}^{\infty} \Delta_j = \Gamma_0$. For more details, see A.5.1 and A.5.2 of the Appendix.

7.6 Suppose one is given a grouping $\mathcal{D} = (\Delta_j)_{j \in \mathbb{N}}$ covering Γ_0 and satisfying (7.6). As is described in § 10, one may construct polynomials $q_{p_j,v}$ in two indeterminates over the real field (v being a suitable fixed integer not less than 36 and p_j any positive number not less than $||D_{\Delta_j}||_{\infty}$) such that, for suitable unimodular complex numbers ξ_j , the t.p.s

$$Q_{j} = \xi_{j} \left(1 + \frac{1}{\nu} \right)^{-1} q_{p_{j},\nu} \left(D_{\Delta_{j}}, \overline{D}_{\Delta_{j}} \right)$$

satisfy

$$||Q_j|| \le 1, \, sp(Q_j) \subseteq [\Delta_j] \subseteq \Gamma_0,$$

$$S_{A_j} Q_j(0) = \int_G D_{A_j} Q_j \, d\lambda_G \text{ is real and } \ge \frac{1}{2} \rho_j.$$

$$(7.7)$$

In view of (7.2), (7.6) and (7.7), one may choose inductively a sequence $(j_n)_{n\in\mathbb{N}}$ of positive integers so that

$$S_{A_{j_n}} Q_{j_n}(0) \text{ is real and } > n^3,$$

$$j_n < j_{n+1}, \ sp(Q_{j_n}) \subseteq \Gamma_0.$$
(7.8)

Accordingly, the t.p.s

$$u_n = n^{-2} Q_{j_n}$$

satisfy the conditions

$$sp (u_n) \subseteq \Gamma_0, \sum_{n=1}^{\infty} || u_n || < \infty$$

$$S_{\Delta_{j_n}} u_n (0) \text{ is real and } > n.$$
(7.9)

At this point the construction in § 2 will yield integers $0 < n_1 < n_2 < ...$ and specifiable sequences $(\gamma_p)_{p \in N}$ of positive numbers such that each function of the form

$$f = \sum_{p=1}^{\infty} \gamma_p \, u_{n_p}$$

is continuous and satisfies

$$sp(f) \subseteq \Gamma_0, \lim_{p \to \infty} \operatorname{Re} S_{A_{j_{n_p}}} f(0) = \infty.$$
 (7.10)

A fortiori, f satisfies (7.3).

We add here that, if the Δ_j are symmetric, the D_{Δ_j} are real-valued, and we may work throughout with real-valued functions, replacing Re $S_{\Delta_j} f$ by $S_{\Delta_j} f$ everywhere.

§ 8. Discussion of case (i): G not 0-dimensional

8.1 In this case $\Phi \neq \Gamma$, and we begin by considering a finite subset of Γ of the form \cdot

$$\Delta = \Omega + \Lambda, \tag{8.1}$$

where Ω and Λ are finite subsets of Γ such that $\pi \mid \Omega$ is 1-1 and $\emptyset \neq \Lambda \subseteq \Phi$. We aim to show that (for a suitable absolute constant k > 0)

$$||D_{\Delta}||_{1} \ge k \left(\frac{\log N}{\log \log N}\right)^{\frac{1}{4}}, \tag{8.2}$$

provided $N = |\Omega|$ (the cardinal number of Ω) is sufficiently large.

8.2 PROOF OF (8.2). Introduce H as the annihilator in G of Φ and identify in the usual way the dual of H with Γ/Φ . Likewise identify the dual of K = G/H with Φ ([7], (24.11)).