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PRINCIPLES, WITH APPLICATIONS TO HARMONIC ANALYSIS

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satisfying $1 \le p < 2 < q \le \infty$, the series (6.6) converges normally in $L_p^q(G)$ to T. Next, T is the limit in E of

$$S_r = \sum_{n=1}^r \omega_n \, T_{K_n}$$

as $r \to \infty$ and, since it is plain that supp $S_r \subseteq \Omega$ for every r, (ii) is easily derived. Finally, if \hat{T} were a measure μ , it would necessarily be the case that supp $\mu \subseteq \overline{\Omega}$ and so, for every $n \in N$, one would have by (6.1) and (6.4)

$$f_n(T) = |u_n * Tv_n(0)| = |\int_{\Gamma} \hat{u}_n \hat{v}_n d\mu|$$

$$\leq |\mu|(\overline{\Omega}),$$

which is finite since Ω is relatively compact. However, this plainly would entail $f^*(T) < \infty$, in conflict with (6.8), so that T cannot be a measure and (iii) is verified. This completes the proof.

6.4 Remark. Theorem 6.3 was proved by Hörmander ([14], Theorem 1.9) for $G = R^n$ and any given pair (p, q) satisfying $1 \le p < 2 < q \le \infty$, this result being extended to a general noncompact LCA G by Gaudry [5]. The argument given by Hörmander (loc. cit. Theorem 1.6 and the remark immediately following) for the case $G = R^n$ can also be extended to a general LCA G and shows that, if either $q \le 2$ or $p \ge 2$, then every $T \in L_{p}^{q}(G)$ is such that \hat{T} is a measure [and indeed a measure of the form $\psi \lambda_{\Gamma}$, where $\psi \in L_{loc}^{2}(\Gamma)$ if $q \le 2$ and $\psi \in L_{loc}^{p}(\Gamma)$ if $p \ge 2$, and so $\psi \in L_{loc}^{2}(\Gamma)$ in either case]. Thus the hypotheses made in Theorem 6.3 about p and q are necessary for the validity of the conclusion.

PART 3: APPLICATIONS TO FOURIER SERIES

§ 7. Applications to divergence of Fourier series.

7.1 Throughout §§ 7-10, G will denote an infinite Hausdorff compact Abelian group with character group Γ , and λ_G the Haar measure on G, normalised so that $\lambda_G(G) = 1$. For any $f \in L^1(G)$, \hat{f} will denote the Fourier transform of f; for any finite subset Δ of Γ ,

$$S_{\Delta}f = \sum_{\gamma \in \Delta} \hat{f}(\gamma)\gamma \tag{7.1}$$

is the Δ -partial sum of the Fourier series of f; and sp (f) will stand for

the spectrum of f, i.e., for the support supp $\hat{f} = \{ \gamma \in \Gamma : \hat{f}(\gamma) \neq 0 \}$ of \hat{f} . The term "trigonometric polynomial" will frequently be abbreviated to "t.p.". In addition, Φ will denote the largest torsion subgroup of Γ ([7], (A.4)), and π the natural map of Γ onto Γ/Φ . If Δ denotes a subset of Γ , $[\Delta]$ will stand for the subgroup of Γ generated by Δ .

By a (convergence) grouping we shall mean a sequence $\mathcal{D} = (\Delta_j)_{j \in \mathbb{N}} = (\Delta_j)$ of finite subsets Δ_j of Γ such that

$$\Delta_{j} \subseteq \Delta_{j+1} \quad (j \in N);$$

$$\overset{\circ}{\bigcup} \Delta_{j} = \Gamma_{0} \text{ is a subgroup of } \Gamma, \text{ said to be}$$

$$covered \ by \ \mathcal{D};$$
for each $j \in N$, $\Delta_{j} = \Omega_{j} + \Lambda_{j}$, where Λ_{j} is a nonvoid finite subset of Φ and Ω_{j} is a finite subset of Γ such that $\pi \mid \Omega_{j}$ is 1-1.

[The first two conditions are natural enough in the context described in 7.3, but the third is less so and may well be pointless.] The grouping \mathcal{D} is said to be of *infinite type* if and only if $\pi(\Gamma_0)$ is infinite.

- 7.2 EXAMPLES. (i) Let Γ_0 be any countable subgroup of Γ such that $\Gamma_0 \cap \Phi = \{0\}$; for example, $\Gamma_0 = \{n\gamma_0 : n \in Z\}$, where $\gamma_0 \in \Gamma \setminus \Phi$. Then a grouping $\mathcal D$ covering Γ_0 results whenever $\Lambda_j = \{0\}$ and $\Delta_j = \Omega_j$ for every $j \in N$, where $(\Omega_j)_{j \in N}$ is any increasing sequence of finite subsets of Γ_0 with union equal to Γ_0 . This grouping is of infinite type if and only if Γ_0 is infinite.
- (ii) If G is connected, and if Γ_0 is any countable subgroup of Γ , then ([10], 2.5.6 (c), 8.1.2 (a) and (b) and 8.1.6) Γ_0 is an ordered group isomorphic to a discrete subgroup of R. Assuming $\Gamma_0 \neq \{0\}$, Γ_0 has a smallest positive element γ_0 and $\Gamma_0 = \{n\gamma_0 : n \in Z\}$. A natural grouping \mathcal{D} covering Γ_0 is that in which $\Lambda_j = \{0\}$ and

$$\Delta_j = \Omega_j = \{ n\gamma_0 : n \in \mathbb{Z}, \mid n \mid \leq j \}$$

for every $j \in N$; this grouping is of infinite type.

7.3 A grouping $\mathscr{D} = (\Delta_j)_{j \in \mathbb{N}}$ will be thought of as specifying one of the many possible ways in which one may interpret the convergence of Fourier series of functions f on G satisfying $sp(f) \subseteq \Gamma_0$, namely, as convergence of the corresponding sequence of partial sums $(S_{\Delta_j} f)_{j \in \mathbb{N}}$.

Indeed, the conditions (7.2) guarantee that $\lim_{j\to\infty} S_{\Delta_j} f = f$ for all sufficiently regular such functions f. However, our concern rests with the possibility of constructing continuous functions f on G satisfying

$$\operatorname{sp}(f) \subseteq \Gamma_0, \overline{\lim_{j \to \infty}} \operatorname{Re} S_{\Delta_j} f(0) = \infty. \tag{7.3}$$

It will appear that the possibilities exhibit a fairly clear dichotomy, depending largely upon whether G is or is not 0-dimensional.

In the first place, it will emerge in 7.6 that the construction principle of § 2, applied to the Banach space E = C(G) of continuous complex valued functions on G [with norm $||\cdot||$ equal to the maximum modulus] and to sequences of gauges of the type

$$f \mid \to \operatorname{Re} S_{\Delta} f(0) = \operatorname{Re} \int_{G} D_{\Delta} f d\lambda_{G},$$
 (7.4)

where D_A stands for the "Dirichlet function"

$$D_{\Delta} = \sum_{\gamma \in \Delta} \bar{\gamma},\tag{7.5}$$

shows that the problem hinges on the existence of groupings \mathcal{D} for which

$$\rho_j = \| D_{\Delta_j} \|_1 = \int_G | D_{\Delta_j} | d\lambda_G \to \infty.$$
 (7.6)

Accordingly, and in view of the fact ([7], (24.26)) that G is 0-dimensional if and only if Γ coincides with Φ , it emerges that the dichotomy referred to may be expressed in the following way.

7.4 Two cases arise, namely:

- (i) G is not 0-dimensional (i.e., $\Phi \neq \Gamma$). Then (see Example 7.2 (i)) there exist groupings $\mathscr{D} = (\Delta_j)$ of infinite type; and, for any such grouping, one can construct (fairly explicitly, as described in 7.6) continuous functions f on G satisfying (7.3). In particular [cf. Example 7.2 (i)], if Γ_0 is any countably infinite subgroup of Γ satisfying $\Gamma_0 \cap \Phi = \{0\}$, and if $(\Delta_j)_{j \in N}$ is any increasing sequence of finite subsets of Γ_0 with union Γ_0 , we can construct a continuous f on G satisfying (7.3).
- (ii) G is 0-dimensional (i.e., $\Phi = \Gamma$). Then there exists no grouping of infinite type. However, given any countable subgroup Γ_0 of Γ , there are groupings $\mathcal{D} = (\Delta_j)$ covering Γ_0 , in which $\Omega_j = \{0\}$ and $\Delta_j = \Lambda_j$ is a finite subgroup of Γ_0 , and for which

$$f = \lim_{i \to \infty} S_{A_j} f$$

uniformly on G for every continuous f satisfying sp $(f) \subseteq \Gamma_0$.

Case (i) will be dealt with in § 8, case (ii) in § 9. The groupings described in case (ii) prove to be exceptional in various ways; see 9.3.

7.5 REMARK. Perhaps it should be stressed here that, if Γ_0 is any infinite subgroup of Γ , there is no obstacle to constructing continuous functions f such that $\operatorname{sp}(f) \subseteq \Gamma_0$ and finite subsets $\Delta_j \subseteq \Delta_{j+1}$ of Γ_0 for which

$$\lim_{i} S_{A_{j}} f(0) = \infty.$$

[One has in fact only to construct a continuous f such that $\operatorname{sp}(f) \subseteq \Gamma_0$ and $\sum_{\gamma \in \Gamma} |\hat{f}(\gamma)| = \infty$; it is then trivial that there exist finite subsets Δ of Γ_0 for which $|S_{\Delta}f(0)|$ is arbitrarily large, so that we can choose a sequence (Δ_j) for which $\Delta_j \subseteq \Delta_{j+1}$ and $|S_{\Delta_j}f(0)| \to \infty$ with j.] However, the sets Δ_j obtained this way will not [and, in view of 7.4 (ii), cannot] in general be such that $\bigcup_{j=1}^{\infty} \Delta_j = \Gamma_0$. For more details, see A.5.1 and A.5.2 of the Appendix.

7.6 Suppose one is given a grouping $\mathcal{D} = (\Delta_j)_{j \in \mathbb{N}}$ covering Γ_0 and satisfying (7.6). As is described in § 10, one may construct polynomials $q_{p_j,v}$ in two indeterminates over the real field (v being a suitable fixed integer not less than 36 and p_j any positive number not less than $||D_{\Delta_j}||_{\infty}$) such that, for suitable unimodular complex numbers ξ_j , the t.p.s

$$Q_{j} = \xi_{j} \left(1 + \frac{1}{\nu} \right)^{-1} q_{p_{j},\nu} \left(D_{\Delta_{j}}, \overline{D}_{\Delta_{j}} \right)$$

satisfy

$$||Q_j|| \le 1, \, sp(Q_j) \subseteq [\Delta_j] \subseteq \Gamma_0,$$

$$S_{\Delta_j} Q_j(0) = \int_G D_{\Delta_j} Q_j \, d\lambda_G \text{ is real and } \ge \frac{1}{2} \rho_j.$$

$$(7.7)$$

In view of (7.2), (7.6) and (7.7), one may choose inductively a sequence $(j_n)_{n\in\mathbb{N}}$ of positive integers so that

$$S_{A_{j_n}} Q_{j_n}(0) \text{ is real and } > n^3,$$

$$j_n < j_{n+1}, \ sp(Q_{j_n}) \subseteq \Gamma_0.$$
(7.8)

Accordingly, the t.p.s

$$u_n = n^{-2} Q_{j_n}$$

satisfy the conditions

$$sp (u_n) \subseteq \Gamma_0, \sum_{n=1}^{\infty} || u_n || < \infty$$

$$S_{A_{j_n}} u_n (0) \text{ is real and } > n.$$
(7.9)

At this point the construction in § 2 will yield integers $0 < n_1 < n_2 < ...$ and specifiable sequences $(\gamma_p)_{p \in N}$ of positive numbers such that each function of the form

$$f = \sum_{p=1}^{\infty} \gamma_p \, u_{n_p}$$

is continuous and satisfies

$$sp(f) \subseteq \Gamma_0, \lim_{p \to \infty} \operatorname{Re} S_{A_{j_{n_p}}} f(0) = \infty.$$
 (7.10)

A fortiori, f satisfies (7.3).

We add here that, if the Δ_j are symmetric, the D_{Δ_j} are real-valued, and we may work throughout with real-valued functions, replacing Re $S_{\Delta_j} f$ by $S_{\Delta_j} f$ everywhere.

§ 8. Discussion of case (i): G not 0-dimensional

8.1 In this case $\Phi \neq \Gamma$, and we begin by considering a finite subset of Γ of the form \cdot

$$\Delta = \Omega + \Lambda, \tag{8.1}$$

where Ω and Λ are finite subsets of Γ such that $\pi \mid \Omega$ is 1-1 and $\emptyset \neq \Lambda \subseteq \Phi$. We aim to show that (for a suitable absolute constant k > 0)

$$||D_{\Delta}||_{1} \ge k \left(\frac{\log N}{\log \log N}\right)^{\frac{1}{4}}, \tag{8.2}$$

provided $N = |\Omega|$ (the cardinal number of Ω) is sufficiently large.

8.2 PROOF OF (8.2). Introduce H as the annihilator in G of Φ and identify in the usual way the dual of H with Γ/Φ . Likewise identify the dual of K = G/H with Φ ([7], (24.11)).

We then have

$$||D_{A}||_{1} = \int_{G} |\sum_{\gamma \in A} \gamma | d\lambda_{G}$$

$$= \int_{G/H} d\lambda_{G/H}(\bar{x}) \int_{H} |\sum_{\theta \in \Omega} \sum_{\phi \in A} \theta (x+y) \phi (x+y) | d\lambda_{H}(y),$$

the inner integral being viewed as a function of $\bar{x} = x + H$ Thus, writing $\bar{\theta}$ for $\pi(\theta)$ and noting that $\phi(y) = 1$ for $\phi \in \Lambda \subseteq \Phi$ and $y \in H$, we obtain

$$\|D_{\Delta}\|_{1} = \int_{G/H} d\lambda_{G/H}(\bar{x}) \int_{H} |\sum_{\theta \in \Omega} \alpha(\theta, x) \bar{\theta}(y)| d\lambda_{H}(y), \tag{8.3}$$

where

$$\alpha(\theta, x) = \theta(x) \sum_{\phi \in A} \phi(x).$$

Now, since the dual of H (namely Γ/Φ) is torsion-free ([7], (A.4)), Theorem A of [8] shows that (for a suitable absolute constant k > 0) we have

$$\int_{H} \left| \sum_{\theta \in \Omega} \alpha \left(\theta, x \right) \overline{\theta} \left(y \right) \right| d\lambda_{H}(y) \ge k \left(\frac{\log N}{\log \log N} \right)^{\frac{1}{4}} \min_{\theta \in \Omega} \left| \alpha \left(\theta, x \right) \right| \\
= k \left(\frac{\log N}{\log \log N} \right)^{\frac{1}{4}} \left| \sum_{\phi \in A} \phi \left(\overline{x} \right) \right|, \tag{8.4}$$

since $|\theta(x)| = 1$ and $\phi(x)$ depends only \bar{x} . By (8.3) and (8.4),

$$||D_{\Delta}||_{1} \geq k \left(\frac{\log N}{\log \log N}\right)^{\frac{1}{4}} \int_{G/H} |\sum_{\phi \in \Lambda} \phi(\overline{x})| d\lambda_{G/\Pi}(\overline{x}). \tag{8.5}$$

Since $\Lambda \neq \emptyset$, the remaining integral is not less than the maximum modulus of the Fourier transform of the function $\bar{x} \mid \to \sum_{\phi \in \Lambda} \phi(\bar{x})$, i.e., is not less than unity. Thus, (8.2) follows from (8.5).

- 8.3 PROOF OF 7.4 (i). The conclusions stated in case (i) of 7.4 are now almost immediate. If $\mathcal{D} = (\Delta_j)_{j \in \mathbb{N}}$ is a grouping of infinite type covering Γ_0 , $|\pi(\Delta_j)| \to \infty$ and so, since $\Lambda_j \subseteq \Phi$, $|\pi(\Omega_j)| \to \infty$. Then (8.2) shows that (7.6) is satisfied, and it remains only to refer to 7.6.
- 8.4 SUPPLEMENTARY REMARKS. The fact that, when G is not 0-dimensional, (7.6) holds for suitable subgroups Γ_0 of Γ and suitable groupings $\mathcal{D} = (\Delta_j)_{j \in \mathbb{N}}$ covering Γ_0 can be derived without appeal to Theorem A

of [8]. To do this, it suffices to take $\gamma_k \in \Gamma \setminus \Phi$ (k = 1, 2, ..., m) such that the family $(\gamma_k)_{1 \le k \le m}$ is independent (see [7], (A.10)), define

$$\Gamma_0 = \{ \sum_{k=1}^m n_k \, \gamma_k : n_k \in Z \text{ for } k = 1, 2, ..., m \},$$

and make use of the formula

$$\int_{G} F(\gamma_{1}(x), ..., \gamma_{m}(x)) d\gamma_{G}(x)$$

$$= (2\pi)^{-m} \int_0^{2\pi} \dots \int_0^{2\pi} F(e^{it}, \dots, e^{it_m}) dt_1 \dots dt_m, \tag{8.6}$$

valid for every $F \in C(T^m)$, where T denotes the circle group. (Recall that $\sum_{k=1}^{m} n_k \gamma_k$ denotes the character $x \mapsto \gamma_1(x)^{n_1} \dots \gamma_m(x)^{n_m}$ of G.) It then appears that (7.6) holds when one takes

$$\Delta_j = \{ \sum_{k=1}^m n_k \gamma_k : |n_k| \le r_{j,k} \text{ for } k = 1, 2, ..., m \},$$

where the $r_{j,k}$ are positive integers satisfying $r_{j,k} \leq r_{j,k+1}$ and $\lim_{j\to\infty} r_{j,k} = \infty$. Moreover, when m=1, the Cohen-Davenport result (essentially Theorem A of [8] for the case G=T) shows that (7.6) holds for every grouping \mathscr{D} covering Γ_0 .

The verification of (8.6) is simple. First note that, if G and G' are compact groups, and if ϕ is a continuous homomorphism of G into G', then

$$\int_{G} (F \circ \phi) d\lambda_{G} = \int F d\lambda_{\phi(G)}$$
 (8.7)

for every $F \in C(G')$. (This is a consequence of the fact that $F \mid \to \int_G (F \circ \phi) d\lambda_G$ is invariant under translation by elements of $\phi(G)$, combined with the uniqueness of the normalised Haar measure on a compact group.) Taking $G' = T^m$ and $\phi: x \mid \to (\gamma_1(x), ..., \gamma_m(x))$, the stated conditions on the γ_k are just adequate to ensure that the annihilator in Z^m (identified in the canonical fashion with the dual of T^m) of $\phi(G)$ is $\{(0, ..., 0)\}$ and so ([7], (24.10)) that $\phi(G) = T^m$. Accordingly, (8.6) appears as a special case of (8.7).

It is perhaps worth indicating that special cases of (8.7) can be exploited in other ways. For example, suppose more generally that κ is an arbitrary nonvoid set and that $(\gamma_k)_{k \in \kappa}$ is a finite or infinite independent family of elements of $\Gamma \setminus \Phi$. Denote by Γ_0 the subgroup of Γ generated by $\{\gamma_k : k \in \kappa\}$. Taking $G' = T^{\kappa}$ and $\phi : x \mid \rightarrow (\gamma_k(x))_{k \in \kappa}$, one may use (8.7) in a similar fashion to show that there is an isometric isomorphism $F \leftrightarrow F \circ \phi = f$ between $L^p(T^{\kappa})$ (or $C(T^{\kappa})$) and the subspace of $L^p(G)$ (or C(G)) formed of those $f \in L^p(G)$ or C(G) such that sp $(f) \subseteq \Gamma_0$. Moreover, if one identifies in the canonical fashion the dual of T^{κ} with the weak

direct product Z^{κ} , the said isomorphism is such that $\hat{F} = \hat{f} \circ \phi'$, where ϕ' is the isomorphism of Z^{κ} onto Γ_0 defined by $(n_k) \to \sum_{k \in \kappa} n_k \gamma_k$.

One consequence of this may be expressed roughly as follows: If the compact Abelian group G is such that $\Gamma \setminus \Phi$ contains an independent family of (finite or infinite) cardinality m, then Fourier series on G behave, in respect of convergence or summability, no better than do Fourier series on T^m .

Another consequence is that, if Δ is a subset of Γ_0 , then Δ is a Sidon (or $\Lambda(p)$) subset of Γ if and only if ${\phi'}^{-1}(\Delta)$ is a Sidon (or $\Lambda(p)$) subset of ${Z^{\kappa}}^*$.

8.5 FURTHER RESULTS. Theorem A of [8] implies something stronger than (8.2), namely: if ω is any complex-valued function on Γ such that

$$\omega(\gamma + \phi) = \omega(\gamma) \quad (\gamma \in \Gamma, \, \phi \in \Phi), \tag{8.8}$$

so that ω can be regarded as a function on Γ/Φ , and if we write

$$D_{\Delta}^{\omega} = \sum_{\gamma \in \Delta} \omega(\gamma) \,\bar{\gamma}, \, S_{\Delta}^{\omega} f = \sum_{\gamma \in \Delta} \omega(\gamma) \,\hat{f}(\gamma), \tag{8.9}$$

then, for $\Delta = \Omega + \Lambda$ as in (8.1), we have

$$||D_{\Delta}^{\omega}||_{1} \ge k \left(\frac{\log N}{\log \log N}\right)^{\frac{1}{4}} \min_{\gamma \in \Omega} |\omega(\gamma)|$$
 (8.10)

provided $N = |\Omega|$ is sufficiently large.

So, if we can arrange for $\Omega = \Omega_j$ to vary in such a way that the right-hand side of (8.10) tends to infinity with j, the substance of 7.6 will lead to a continuous f satisfying sp $(f) \subseteq \Gamma_0$ and

$$\overline{\lim}_{j \to \infty} \operatorname{Re} S_{\Delta_j}^{\omega} f(0) = \infty. \tag{8.11}$$

Taking the most familiar case, in which G = T, $\Gamma = Z$ and $\Phi = \{0\}$, and supposing $\Delta = \Omega$ to range over a sequence (Δ_j) of finite subsets of Z such that, if $N_j = |\Delta_j|$,

$$\lim_{j} \left(\frac{\log N_{j}}{\log \log N_{j}} \right)^{\frac{1}{4}} \min_{n \in A_{j}} |\omega(n)| = \infty,$$

the construction will lead to a continuous f on T such that

$$\lim_{j} \operatorname{Re} S_{\Delta_{j}}^{\omega} f(0) = \infty.$$

In particular, taking $\Delta_j = \{n \in \mathbb{Z} : 2^j \le n < 2^{j+1}\}$ it can be arranged that

$$\sum_{n \in \mathbb{Z}} \frac{\pm \hat{f}(n)}{(\log (2 + |n|))^{\alpha}}$$

diverges for any preassigned distribution of signs \pm and any preassigned $\alpha < \frac{1}{4}$.

Of course, much stronger results are derivable by using random (and unspecifiable!) changes of sign, but there seems little hope of making this even remotely constructive.

§ 9. Discussion of case (ii): G 0-dimensional

9.1 In this case there is ([7], (7.7)) a base of neighbourhoods of zero in G formed of compact open subgroups W. For each such W the annihilator $\Delta = W^{\circ}$ in Γ of W is a finite subgroup of Γ . Define

$$k_W = \lambda_G(W)^{-1} \times \text{characteristic function of } W.$$
 (9.1)

Then k_W is continuous, $k_W \ge 0$, $\int_G k_W d\lambda_G = 1$. The transform \hat{k}_W of k_W is plainly equal to unity on Δ . On the other hand, since W is a subgroup, we have for $a \in W$ and $\gamma \in \Gamma$

$$\hat{k}_{W}(\gamma) = \int_{G} k_{W}(x) \overline{\gamma(x)} d\lambda_{G}(x) = \int_{G} k_{W}(x+a) \overline{\gamma(x)} d\lambda_{G}(x)$$

$$= \int_{G} k_{W}(y) \overline{\gamma(y-a)} d\lambda_{G}(y)$$

$$= \gamma(a) \hat{k}_{W}(y),$$

which shows that $\hat{k}_{W}(\gamma) = 0$ if $\gamma \in \Gamma \setminus \Delta$. Thus \hat{k}_{W} is the characteristic function of Δ , and so

$$k_{\mathbf{W}} = D_{\mathbf{W}^{\circ}}. \tag{9.2}$$

By (9.1) and (9.2), a routine argument shows that, if $1 \le p < \infty$ and $f \in L^p(G)$, then

$$f = \lim_{W} S_{W^{\circ}} f \tag{9.3}$$

in $L^p(G)$; and that (9.3) holds uniformly for any continuous f.

9.2 Proof of 7.4 (ii). If Γ_0 is any countably infinite subgroup of Γ we can choose a sequence W_i of compact open subgroups of G such that

 $W_{j+1} \subseteq W_j$ and $\Gamma_0 \subseteq \bigcup_{j=1}^{\infty} W_j^{\circ}$, where W_j° is a finite subgroup of Γ and $W_j^{\circ} \subseteq W_{j+1}^{\circ}$. The $\Delta_j = W_j^{\circ} \cap \Gamma_0$ satisfy (7.2) and, from (9.3),

$$f = \lim_{j} S_{\Delta_{j}} f \tag{9.4}$$

uniformly for any continuous f with $sp(f) \subseteq \Gamma_0$. This verifies the statements made in 7.4 (ii).

9.3 By using the results in [3], more can be said in case (ii) of 7.4; cf. [3], Theorem (2.9) and Example (4.8).

Let $f \in L^1(G)$ and let Γ_0 be any countable subgroup of Γ containing sp (f). Choose the W_j as in 9.2. Then, apart from the fact that (W_j) is not in general a base at 0 in G (they can be chosen to be so if and only if G is first countable), (W_j) is an open-compact D''-sequence ([3], p. 188). The proof of Theorem (2.5) of [3] is easily modified to show that

$$f(x) = \lim_{j \to \infty} S_{W_j^{\circ}} f(x) \tag{9.5}$$

holds for almost all $x \in G$. Moreover, Theorem (2.7) of [3] applies to show that the majorant function

$$S^* f(x) = \sup_{j \in \mathbb{N}} |S_{W_j^{\circ}} f(x)|$$
 (9.6)

satisfies the estimates

$$||S^*f||_p \le 2(p(p-1)^{-1})^{\frac{1}{p}}||f||_p \quad (1 (9.7)$$

$$||S^*f||_1 \le 2 + 2 \int_G |f| \log^+ |f| d\lambda_G,$$
 (9.8)

$$||S^*f||_p \le 2(1-p)^{\frac{1}{p}}||f||_1 \quad (0 (9.9)$$

In particular, the convergence in (9.5) is dominated whenever

$$|f|\log^+|f| \in L^1(G).$$

A more immediate consequence of (9.1) and (9.2) is a strong version of localisability of the convergence of Fourier series: if $f \in L^1(G)$ vanishes a.e. on some neighbourhood of $x_0 \in G$, we can choose the W_j so that $S_{\Delta_j} f(x_0) = 0$ for every sufficiently large j. [A suitable choice of W_j may be made once for all, independent of f, if G is first countable.] Nothing similar is true for general G; see, for example, [11], Vol. II, pp. 304-305.

§ 10. Concerning the polynomials Q_i .

There is no difficulty in making fairly explicit the construction of t.p.s Q_j of the type employed in 7.6.

For p > 0, $t \ge 0$ define

$$h_{p}(t) = \begin{cases} 1 & \text{if } t \leq p, \\ 2\left(1 - \frac{t}{2p}\right) & \text{if } p \leq t \leq 2p, \\ 0 & \text{if } t \geq 2p. \end{cases}$$
 (10.1)

For all complex z define

$$f_{p}(z) = \begin{cases} 0 & \text{if } z = 0, \\ |z|^{-1} \bar{z} h_{p}(|z|) & \text{if } z \neq 0. \end{cases}$$
 (10.2)

Write

$$E_{n}(z) = \pi^{-1} n \exp(-n |z|^{2},$$

$$P_{n,k}(z) = \pi^{-1} n \sum_{j=0}^{k} \frac{(-1)^{j}}{j!} (n|z|^{2})^{j}$$
(10.3)

Let μ denote Lebesgue measure on C (identified with R^2 in the canonical fashion).

It is then routine to verify that

$$\left\| E_{n} * f_{p} \right\|_{\infty} \leq \left\| f_{p} \right\|_{\infty} = 1,$$

$$\lim_{n \to \infty} E_{n} * f_{p} = f_{p}$$

$$(10.4)$$

uniformly on any compact set omitting 0. From this it follows that to every p > 0 and every positive integer v correspond positive integers $\bar{n}(p, v)$, $\bar{k}(p, v)$ such that

$$\left| \left| z \right|^{-1} \overline{z} - f_{p} * P_{\overline{n}, \overline{k}}(z) \right| \leq \frac{1}{\nu} \text{ for } \frac{1}{\nu} \leq \left| z \right| \leq p,$$

$$\left| f_{p} * P_{\overline{n}, \overline{k}}(z) \right| \leq 1 + \frac{1}{\nu} \text{ for } \left| z \right| \leq p.$$

$$(10.5)$$

Now

$$f_p * P_{\bar{n}, \bar{k}}(z) = q_{p, \nu}(z, \bar{z}),$$
 (10.6)

where

$$q_{p,v}(X, Y) = \pi^{-1} \bar{n}(p, v) \sum_{j=0}^{\bar{k}(p,v)} \frac{(-\bar{n}(p, v))^{j}}{j!} \sum_{l=0}^{j} \sum_{m=0}^{j} {j \choose l} {j \choose m} X^{l} Y^{m}$$

$$(-1)^{l+m} \int \zeta^{j-l} \bar{\zeta}^{j-m} f_{p}(\zeta) d\mu(\zeta)$$

$$\bar{k}(p,v) \int_{\Sigma} C_{-j}(l, m) Y^{l} Y^{m}$$
(10.7)

$$= \sum_{l,m=0}^{\bar{k}(p,v)} C_{p,v}(l,m) X^{l} Y^{m}.$$
 (10.7)

It is easily verifiable that the $C_{p,v}(l,m)$ are real-valued.

If θ is a bounded measurable function on G and

$$Q_{p,\nu}^{\circ} = q_{p,\nu}(\theta, \bar{\theta}), p \ge \|\theta\|_{\infty}, \tag{10.8}$$

we have from (10.5)

$$\left| |\theta|^{-1} \bar{\theta} - Q_{p,v}^{\circ} \right| \leq \frac{1}{v} \text{ whenever } |\theta| \geq \frac{1}{v},$$

$$|Q_{p,v}^{\circ}| \leq 1 + \frac{1}{v} \text{ everywhere on } G.$$
(10.9)

If θ is a t.p., then $Q_{p,v}^{\circ}$ is a t.p. and

$$\operatorname{sp}\left(Q_{p,\nu}^{\circ}\right)\subseteq\left[\operatorname{sp}\left(\theta\right)\right].\tag{10.10}$$

From (10.9) we obtain

$$\left| \left| \theta \right| - \theta \left| Q_{p,v}^{\circ} \right| \leq \begin{cases} v^{-1} \left| \theta \right| \text{ whenever } \left| \theta \right| > \frac{1}{v}, \\ \left(2 + \frac{1}{v} \right) \left| \theta \right| \text{ everywhere,} \end{cases}$$

whence it follows that, if $\theta \neq 0$,

$$\left| \int_{G} \theta \ Q_{p,\nu}^{\circ} d\lambda G \right| \ge (1 - \nu^{-1}) \left\| \theta \right\|_{1} - \nu^{-1} (2 + \nu^{-1})$$

$$\ge (1 - 2\nu^{-\frac{1}{2}}) \left\| \theta \right\|_{1}$$
(10.11)

provided $v \ge 9 \|\theta\|_1^{-2}$.

Taking $\theta = D_{A_j}$ and $p_j \ge ||D_{A_j}||$, the trigonometric polynomials

$$Q'_{j} = \left(1 + \frac{1}{\nu}\right)^{-1} Q^{\circ}_{p_{j},\nu} = \left(1 + \frac{1}{\nu}\right)^{-1} q_{p_{j},\nu} \left(D_{A_{j}}, \overline{D}_{A_{j}}\right) \quad (10.12)$$

are then seen from (10.9), (10.10) and (10.11) to satisfy

$$||Q'_{j}|| \leq 1,$$

$$sp(Q'_{j}) \leq [\Delta_{j}],$$

$$|\int v D_{\Delta_{j}} Q'_{j} d\lambda_{G}| \geq (1 - 3v^{-\frac{1}{2}}) ||D_{\Delta_{j}}||_{1}$$

$$(10.13)$$

provided ν is chosen $\geq 9 \parallel D_{A_j} \parallel_1^{-1}$. In view of (7.6), we may choose the integer $\nu \geq \max_j$ (36, $9 \parallel D_{A_j} \parallel_1^{-1}$). Then (10.13) shows that there are unimodular complex numbers ξ_j such that the $Q_j = \xi_j Q_j'$ satisfy (7.7).

APPENDIX

Rudin-Shapiro sequences

A.1 Notations and definitions. As hitherto, all topological groups G are assumed to be Hausdorff; and, for any locally compact group G, λ_G will denote a selected left Haar measure, with respect to which the Lebesgue spaces $L^p(G)$ are to be formed. $C_c(G)$ denotes the set of complex-valued continuous functions on G having compact supports.

If X and Y are topological groups, Hom (X, Y) denotes the set of continuous homomorphisms of X into Y.

Suppose henceforth G to be locally compact. As in 5.1, if $k \in C_c(G)$, T_k will denote the convolution operator

$$f \mid \rightarrow f * k$$

with domain $C_c(G)$ and range in $C_c(G)$; and $||k||_{p,q}$ will denote the (p,q)-norm of this operator, i.e., the smallest real number $m \ge 0$ such that

$$||f * k||_q \leq m ||f||_p \quad (f \in C_c(G)).$$

It is well-known that, if G is Abelian, $||k||_{2,2}$ is equal to

$$\|\hat{k}\|_{\infty} = \sup_{\gamma \in \Gamma} |\hat{k}(\gamma)|,$$

where Γ is the character group of G and \hat{k} is the Fourier transform of k. (Something similar is true whenever G is compact, but we shall not use this.)

U-RS-sequences on G are as defined in 5.4.