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satisfying $1 \leqq p < 2 < q \leqq \infty$, the series (6.6) converges normally in $L_p^q(G)$ to $T$. Next, $T$ is the limit in $E$ of

$$S_r = \sum_{n=1}^{r} \omega_n T_{K_n}$$

as $r \to \infty$ and, since it is plain that supp $S_r \subseteq \Omega$ for every $r$, (ii) is easily derived. Finally, if $\hat{T}$ were a measure $\mu$, it would necessarily be the case that supp $\mu \subseteq \bar{\Omega}$ and so, for every $n \in N$, one would have by (6.1) and (6.4)

$$\begin{aligned} f_n(T) &= | u_n * Tv_n(0) | = | \textstyle\int_\Gamma \hat{u}_n \hat{v}_n \, d\mu | \\ &\leqq | \mu | (\bar{\Omega}), \end{aligned}$$

which is finite since $\Omega$ is relatively compact. However, this plainly would entail $f^*(T) < \infty$, in conflict with (6.8), so that $T$ cannot be a measure and (iii) is verified. This completes the proof.

6.4 Remark. Theorem 6.3 was proved by Hörmander ([14], Theorem 1.9) for $G = R^n$ and any given pair $(p, q)$ satisfying $1 \leqq p < 2 < q \leqq \infty$, this result being extended to a general noncompact LCA $G$ by Gaudry [5]. The argument given by Hörmander (loc. cit. Theorem 1.6 and the remark immediately following) for the case $G = R^n$ can also be extended to a general LCA $G$ and shows that, if *either* $q \leqq 2$ *or* $p \geqq 2$, then every $T \in L_p^q(G)$ is such that $\hat{T}$ is a measure [and indeed a measure of the form $\psi\lambda_\Gamma$, where $\psi \in L^2_{loc}(\Gamma)$ if $q \leqq 2$ and $\psi \in L^p_{loc}(\Gamma)$ if $p \geqq 2$, and so $\psi \in L^2_{loc}(\Gamma)$ in either case ]. Thus the hypotheses made in Theorem 6.3 about $p$ and $q$ are necessary for the validity of the conclusion.

## Part 3: Applications to Fourier series

### § 7. *Applications to divergence of Fourier series.*

7.1 Throughout §§ 7-10, $G$ will denote an infinite Hausdorff compact Abelian group with character group $\Gamma$, and $\lambda_G$ the Haar measure on $G$, normalised so that $\lambda_G(G) = 1$. For any $f \in L^1(G)$, $\hat{f}$ will denote the Fourier transform of $f$; for any finite subset $\Delta$ of $\Gamma$,

$$S_\Delta f = \sum_{\gamma \in \Delta} \hat{f}(\gamma)\gamma \tag{7.1}$$

is the $\Delta$-partial sum of the Fourier series of $f$; and sp $(f)$ will stand for

the spectrum of $f$, i.e., for the support supp $\hat{f} = \{\gamma \in \Gamma : \hat{f}(\gamma) \neq 0\}$ of $\hat{f}$. The term "trigonometric polynomial" will frequently be abbreviated to "t.p.". In addition, $\Phi$ will denote the largest torsion subgroup of $\Gamma$ ([7], (A.4)), and $\pi$ the natural map of $\Gamma$ onto $\Gamma/\Phi$. If $\Delta$ denotes a subset of $\Gamma$, $[\Delta]$ will stand for the subgroup of $\Gamma$ generated by $\Delta$.

By a *(convergence) grouping* we shall mean a sequence $\mathscr{D} = (\Delta_j)_{j \in N} = (\Delta_j)$ of finite subsets $\Delta_j$ of $\Gamma$ such that

$$\left.\begin{array}{l} \Delta_j \subseteq \Delta_{j+1} \quad (j \in N); \\ \bigcup_{j=1}^{\infty} \Delta_j = \Gamma_0 \text{ is a subgroup of } \Gamma, \text{ said to be} \\ \quad \textit{covered by } \mathscr{D}; \\ \text{for each } j \in N,\ \Delta_j = \Omega_j + \Lambda_j, \text{ where } \Lambda_j \text{ is a} \\ \quad \text{nonvoid finite subset of } \Phi \text{ and } \Omega_j \text{ is a finite} \\ \quad \text{subset of } \Gamma \text{ such that } \pi \mid \Omega_j \text{ is 1-1.} \end{array}\right\} \tag{7.2}$$

[The first two conditions are natural enough in the context described in 7.3, but the third is less so and may well be pointless.] The grouping $\mathscr{D}$ is said to be of *infinite type* if and only if $\pi(\Gamma_0)$ is infinite.

7.2 EXAMPLES. (i) Let $\Gamma_0$ be any countable subgroup of $\Gamma$ such that $\Gamma_0 \cap \Phi = \{0\}$; for example, $\Gamma_0 = \{n\gamma_0 : n \in Z\}$, where $\gamma_0 \in \Gamma \setminus \Phi$. Then a grouping $\mathscr{D}$ covering $\Gamma_0$ results whenever $\Lambda_j = \{0\}$ and $\Delta_j = \Omega_j$ for every $j \in N$, where $(\Omega_j)_{j \in N}$ is any increasing sequence of finite subsets of $\Gamma_0$ with union equal to $\Gamma_0$. This grouping is of infinite type if and only if $\Gamma_0$ is infinite.

(ii) If $G$ is connected, and if $\Gamma_0$ is any countable subgroup of $\Gamma$, then ([10], 2.5.6 (c), 8.1.2 (a) and (b) and 8.1.6) $\Gamma_0$ is an ordered group isomorphic to a discrete subgroup of $R$. Assuming $\Gamma_0 \neq \{0\}$, $\Gamma_0$ has a smallest positive element $\gamma_0$ and $\Gamma_0 = \{n\gamma_0 : n \in Z\}$. A natural grouping $\mathscr{D}$ covering $\Gamma_0$ is that in which $\Lambda_j = \{0\}$ and

$$\Delta_j = \Omega_j = \{n\gamma_0 : n \in Z, |n| \leqq j\}$$

for every $j \in N$; this grouping is of infinite type.

7.3 A grouping $\mathscr{D} = (\Delta_j)_{j \in N}$ will be thought of as specifying one of the many possible ways in which one may interpret the convergence of Fourier series of functions $f$ on $G$ satisfying $sp(f) \subseteq \Gamma_0$, namely, as convergence of the corresponding sequence of partial sums $(S_{\Delta_j} f)_{j \in N}$.

Indeed, the conditions (7.2) guarantee that $\lim_{j\to\infty} S_{\Delta_j} f = f$ for all sufficiently regular such functions $f$. However, our concern rests with the possibility of constructing continuous functions $f$ on $G$ satisfying

$$\mathrm{sp}\,(f) \subseteq \Gamma_0, \ \overline{\lim_{j\to\infty}} \operatorname{Re} S_{\Delta_j} f(0) = \infty. \tag{7.3}$$

It will appear that the possibilities exhibit a fairly clear dichotomy, depending largely upon whether $G$ is or is not 0-dimensional.

In the first place, it will emerge in 7.6 that the construction principle of § 2, applied to the Banach space $E = C(G)$ of continuous complex valued functions on $G$ [with norm $\|\cdot\|$ equal to the maximum modulus] and to sequences of gauges of the type

$$f \mapsto \operatorname{Re} S_\Delta f(0) = \operatorname{Re} \textstyle\int_G D_\Delta f \, d\lambda_G, \tag{7.4}$$

where $D_\Delta$ stands for the "Dirichlet function"

$$D_\Delta = \sum_{\gamma\in\Delta} \bar{\gamma}, \tag{7.5}$$

shows that the problem hinges on the existence of groupings $\mathscr{D}$ for which

$$\rho_j = \| D_{\Delta_j} \|_1 = \textstyle\int_G | D_{\Delta_j} | \, d\lambda_G \to \infty. \tag{7.6}$$

Accordingly, and in view of the fact ([7], (24.26)) that $G$ is 0-dimensional if and only if $\Gamma$ coincides with $\Phi$, it emerges that the dichotomy referred to may be expressed in the following way.

7.4 Two cases arise, namely:

(i) $G$ is not 0-dimensional (i.e., $\Phi \neq \Gamma$). Then (see Example 7.2 (i)) there exist groupings $\mathscr{D} = (\Delta_j)$ of infinite type; and, for any such grouping, one can construct (fairly explicitly, as described in 7.6) continuous functions $f$ on $G$ satisfying (7.3). In particular [cf. Example 7.2 (i)], if $\Gamma_0$ is any countably infinite subgroup of $\Gamma$ satisfying $\Gamma_0 \cap \Phi = \{0\}$, and if $(\Delta_j)_{j\in N}$ is any increasing sequence of finite subsets of $\Gamma_0$ with union $\Gamma_0$, we can construct a continuous $f$ on $G$ satisfying (7.3).

(ii) $G$ is 0-dimensional (i.e., $\Phi = \Gamma$). Then there exists no grouping of infinite type. However, given any countable subgroup $\Gamma_0$ of $\Gamma$, there are groupings $\mathscr{D} = (\Delta_j)$ covering $\Gamma_0$, in which $\Omega_j = \{0\}$ and $\Delta_j = \Lambda_j$ is a finite subgroup of $\Gamma_0$, and for which

$$f = \lim_{j\to\infty} S_{\Delta_j} f$$

uniformly on $G$ for every continuous $f$ satisfying $\mathrm{sp}(f) \subseteq \Gamma_0$.

Case (i) will be dealt with in § 8, case (ii) in § 9. The groupings described in case (ii) prove to be exceptional in various ways; see 9.3.

7.5 REMARK. Perhaps it should be stressed here that, if $\Gamma_0$ is any infinite subgroup of $\Gamma$, there is no obstacle to constructing continuous functions $f$ such that $\mathrm{sp}(f) \subseteq \Gamma_0$ and finite subsets $\Delta_j \subseteq \Delta_{j+1}$ of $\Gamma_0$ for which

$$\lim_j S_{\Delta_j} f(0) = \infty.$$

[One has in fact only to construct a continuous $f$ such that $\mathrm{sp}(f) \subseteq \Gamma_0$ and $\sum_{\gamma \in \Gamma} |\hat{f}(\gamma)| = \infty$; it is then trivial that there exist finite subsets $\Delta$ of $\Gamma_0$ for which $|S_\Delta f(0)|$ is arbitrarily large, so that we can choose a sequence $(\Delta_j)$ for which $\Delta_j \subseteq \Delta_{j+1}$ and $|S_{\Delta_j} f(0)| \to \infty$ with $j$.] However, the sets $\Delta_j$ obtained this way will not [and, in view of 7.4 (ii), cannot] in general be such that $\bigcup_{j=1}^{\infty} \Delta_j = \Gamma_0$. For more details, see A.5.1 and A.5.2 of the Appendix.

7.6 Suppose one is given a grouping $\mathscr{D} = (\Delta_j)_{j \in N}$ covering $\Gamma_0$ and satisfying (7.6). As is described in § 10, one may construct polynomials $q_{p_j,\nu}$ in two indeterminates over the real field ($\nu$ being a suitable fixed integer not less than 36 and $p_j$ any positive number not less than $\| D_{\Delta_j} \|_\infty$) such that, for suitable unimodular complex numbers $\xi_j$, the t.p.s

$$Q_j = \xi_j \left(1 + \frac{1}{\nu}\right)^{-1} q_{p_j,\nu}(D_{\Delta_j}, \bar{D}_{\Delta_j})$$

satisfy

$$\left.\begin{aligned} &\| Q_j \| \leqq 1, \ sp(Q_j) \subseteq [\Delta_j] \subseteq \Gamma_0, \\ &S_{\Delta_j} Q_j(0) = \textstyle\int_G D_{\Delta_j} Q_j \, d\lambda_G \text{ is real and } \geqq \tfrac{1}{2} \rho_j. \end{aligned}\right\} \tag{7.7}$$

In view of (7.2), (7.6) and (7.7), one may choose inductively a sequence $(j_n)_{n \in N}$ of positive integers so that

$$\left.\begin{aligned} &S_{\Delta_{j_n}} Q_{j_n}(0) \text{ is real and } > n^3, \\ &j_n < j_{n+1}, \ sp(Q_{j_n}) \subseteq \Gamma_0. \end{aligned}\right\} \tag{7.8}$$

Accordingly, the t.p.s

$$u_n = n^{-2} Q_{j_n}$$

satisfy the conditions

$$\left.\begin{aligned} &\mathrm{sp}\,(u_n) \subseteq \Gamma_0, \sum_{n=1}^{\infty} \| u_n \| < \infty \\ &S_{\Delta_{j_n}} u_n\,(0) \text{ is real and } > n. \end{aligned}\right\} \tag{7.9}$$

At this point the construction in § 2 will yield integers $0 < n_1 < n_2 < \ldots$ and specifiable sequences $(\gamma_p)_{p \in N}$ of positive numbers such that each function of the form

$$f = \sum_{p=1}^{\infty} \gamma_p u_{n_p}$$

is continuous and satisfies

$$sp\,(f) \subseteq \Gamma_0, \lim_{p \to \infty} \mathrm{Re}\, S_{\Delta_{j_{n_p}}} f(0) = \infty. \tag{7.10}$$

A fortiori, $f$ satisfies (7.3).

We add here that, if the $\Delta_j$ are symmetric, the $D_{\Delta_j}$ are real-valued, and we may work throughout with real-valued functions, replacing $\mathrm{Re}\, S_{\Delta_j} f$ by $S_{\Delta_j} f$ everywhere.

## § 8. *Discussion of case (i) :* G *not* 0-*dimensional*

8.1 In this case $\Phi \neq \Gamma$, and we begin by considering a finite subset of $\Gamma$ of the form

$$\Delta = \Omega + \Lambda, \tag{8.1}$$

where $\Omega$ and $\Lambda$ are finite subsets of $\Gamma$ such that $\pi \mid \Omega$ is 1-1 and $\varnothing \neq \Lambda \subseteq \Phi$. We aim to show that (for a suitable absolute constant $k > 0$)

$$\| D_\Delta \|_1 \geqq k \left( \frac{\log N}{\log \log N} \right)^{\frac{1}{4}}, \tag{8.2}$$

provided $N = | \Omega |$ (the cardinal number of $\Omega$) is sufficiently large.

8.2 Proof of (8.2). Introduce $H$ as the annihilator in $G$ of $\Phi$ and identify in the usual way the dual of $H$ with $\Gamma/\Phi$. Likewise identify the dual of $K = G/H$ with $\Phi$ ([7], (24.11)).

We then have

$$\| D_\Delta \|_1 = \int_G \Big| \sum_{\gamma \in \Delta} \gamma \Big| \, d\lambda_G$$

$$= \int_{G/H} d\lambda_{G/H}(\bar{x}) \int_H \Big| \sum_{\theta \in \Omega} \sum_{\phi \in \Lambda} \theta(x+y)\, \phi(x+y) \Big| \, d\lambda_H(y),$$

the inner integral being viewed as a function of $\bar{x} = x+H$ Thus, writing $\bar{\theta}$ for $\pi(\theta)$ and noting that $\phi(y) = 1$ for $\phi \in \Lambda \subseteq \Phi$ and $y \in H$, we obtain

$$\| D_\Delta \|_1 = \int_{G/H} d\lambda_{G/H}(\bar{x}) \int_H \Big| \sum_{\theta \in \Omega} \alpha(\theta, x) \bar{\theta}(y) \Big| \, d\lambda_H(y), \tag{8.3}$$

where

$$\alpha(\theta, x) = \theta(x) \sum_{\phi \in \Lambda} \phi(x).$$

Now, since the dual of $H$ (namely $\Gamma/\Phi$) is torsion-free ([7], (A.4)), Theorem A of [8] shows that (for a suitable absolute constant $k > 0$) we have

$$\int_H \Big| \sum_{\theta \in \Omega} \alpha(\theta, x) \bar{\theta}(y) \Big| \, d\lambda_H(y) \geqq k \left( \frac{\log N}{\log \log N} \right)^{\frac{1}{4}} \min_{\theta \in \Omega} | \alpha(\theta, x) |$$

$$= k \left( \frac{\log N}{\log \log N} \right)^{\frac{1}{4}} \Big| \sum_{\phi \in \Lambda} \phi(\bar{x}) \Big|, \tag{8.4}$$

since $| \theta(x) | = 1$ and $\phi(x)$ depends only $\bar{x}$. By (8.3) and (8.4),

$$\| D_\Delta \|_1 \geqq k \left( \frac{\log N}{\log \log N} \right)^{\frac{1}{4}} \int_{G/H} \Big| \sum_{\phi \in \Lambda} \phi(\bar{x}) \Big| \, d\lambda_{G/H}(\bar{x}). \tag{8.5}$$

Since $\Lambda \neq \varnothing$, the remaining integral is not less than the maximum modulus of the Fourier transform of the function $\bar{x} \mapsto \sum_{\phi \in \Lambda} \phi(\bar{x})$, i.e., is not less than unity. Thus, (8.2) follows from (8.5).

8.3 Proof of 7.4 (i). The conclusions stated in case (i) of 7.4 are now almost immediate. If $\mathscr{D} = (\Delta_j)_{j \in N}$ is a grouping of infinite type covering $\Gamma_0$, $| \pi(\Delta_j) | \to \infty$ and so, since $\Lambda_j \subseteq \Phi$, $| \pi(\Omega_j) | \to \infty$. Then (8.2) shows that (7.6) is satisfied, and it remains only to refer to 7.6.

8.4 Supplementary remarks. The fact that, when $G$ is not 0-dimensional, (7.6) holds for suitable subgroups $\Gamma_0$ of $\Gamma$ and suitable groupings $\mathscr{D} = (\Delta_j)_{j \in N}$ covering $\Gamma_0$ can be derived without appeal to Theorem A

of [8]. To do this, it suffices to take $\gamma_k \in \Gamma \setminus \Phi$ $(k = 1, 2, ..., m)$ such that the family $(\gamma_k)_{1 \leqq k \leqq m}$ is independent (see [7], (A.10)), define

$$\Gamma_0 = \{\textstyle\sum_{k=1}^m n_k \gamma_k : n_k \in Z \text{ for } k = 1, 2, ..., m\},$$

and make use of the formula

$$\int_G F(\gamma_1(x), ..., \gamma_m(x)) d\gamma_G(x)$$
$$= (2\pi)^{-m} \int_0^{2\pi} ... \int_0^{2\pi} F(e^{it}, ..., e^{it_m}) dt_1 ... dt_m, \tag{8.6}$$

valid for every $F \in C(T^m)$, where $T$ denotes the circle group. (Recall that $\sum_{k=1}^m n_k \gamma_k$ denotes the character $x \mapsto \gamma_1(x)^{n_1} ... \gamma_m(x)^{n_m}$ of $G$.) It then appears that (7.6) holds when one takes

$$\Delta_j = \{\textstyle\sum_{k=1}^m n_k \gamma_k : |n_k| \leqq r_{j,k} \text{ for } k = 1, 2, ..., m\},$$

where the $r_{j,k}$ are positive integers satisfying $r_{j,k} \leqslant r_{j,k+1}$ and $\lim_{j \to \infty} r_{j,k} = \infty$. Moreover, when $m = 1$, the Cohen-Davenport result (essentially Theorem A of [8] for the case $G = T$) shows that (7.6) holds for every grouping $\mathscr{D}$ covering $\Gamma_0$.

The verification of (8.6) is simple. First note that, if $G$ and $G'$ are compact groups, and if $\phi$ is a continuous homomorphism of $G$ into $G'$, then

$$\int_G (F \circ \phi) d\lambda_G = \int F d\lambda_{\phi(G)} \tag{8.7}$$

for every $F \in C(G')$. (This is a consequence of the fact that $F \mapsto \int_G (F \circ \phi) d\lambda_G$ is invariant under translation by elements of $\phi(G)$, combined with the uniqueness of the normalised Haar measure on a compact group.) Taking $G' = T^m$ and $\phi : x \mapsto (\gamma_1(x), ..., \gamma_m(x))$, the stated conditions on the $\gamma_k$ are just adequate to ensure that the annihilator in $Z^m$ (identified in the canonical fashion with the dual of $T^m$) of $\phi(G)$ is $\{(0, ..., 0)\}$ and so ([7], (24.10)) that $\phi(G) = T^m$. Accordingly, (8.6) appears as a special case of (8.7).

It is perhaps worth indicating that special cases of (8.7) can be exploited in other ways. For example, suppose more generally that $\kappa$ is an arbitrary nonvoid set and that $(\gamma_k)_{k \in \kappa}$ is a finite or infinite independent family of elements of $\Gamma \setminus \Phi$. Denote by $\Gamma_0$ the subgroup of $\Gamma$ generated by $\{\gamma_k : k \in \kappa\}$. Taking $G' = T^\kappa$ and $\phi : x \mapsto (\gamma_k(x))_{k \in \kappa}$, one may use (8.7) in a similar fashion to show that there is an isometric isomorphism $F \leftrightarrow F \circ \phi = f$ between $L^p(T^\kappa)$ (or $C(T^\kappa)$) and the subspace of $L^p(G)$ (or $C(G)$) formed of those $f \in L^p(G)$ or $C(G)$) such that $\mathrm{sp}(f) \subseteq \Gamma_0$. Moreover, if one identifies in the canonical fashion the dual of $T^\kappa$ with the weak

direct product $Z^{\kappa^*}$, the said isomorphism is such that $\hat{F} = \hat{f} \circ \phi'$, where $\phi'$ is the isomorphism of $Z^{\kappa^*}$ onto $\Gamma_0$ defined by $(n_k) \to \sum_{k \in \kappa} n_k \gamma_k$.

One consequence of this may be expressed roughly as follows: If the compact Abelian group $G$ is such that $\Gamma \setminus \Phi$ contains an independent family of (finite or infinite) cardinality $m$, then Fourier series on $G$ behave, in respect of convergence or summability, no better than do Fourier series on $T^m$.

Another consequence is that, if $\Delta$ is a subset of $\Gamma_0$, then $\Delta$ is a Sidon (or $\Lambda(p)$) subset of $\Gamma$ if and only if $\phi'^{-1}(\Delta)$ is a Sidon (or $\Lambda(p)$) subset of $Z^{\kappa^*}$.

8.5 Further results. Theorem A of [8] implies something stronger than (8.2), namely: if $\omega$ is any complex-valued function on $\Gamma$ such that

$$\omega(\gamma + \phi) = \omega(\gamma) \quad (\gamma \in \Gamma, \phi \in \Phi), \tag{8.8}$$

so that $\omega$ can be regarded as a function on $\Gamma/\Phi$, and if we write

$$D_\Delta^\omega = \sum_{\gamma \in \Delta} \omega(\gamma) \bar{\gamma}, \; S_\Delta^\omega f = \sum_{\gamma \in \Delta} \omega(\gamma) \hat{f}(\gamma), \tag{8.9}$$

then, for $\Delta = \Omega + \Lambda$ as in (8.1), we have

$$\| D_\Delta^\omega \|_1 \geqq k \left( \frac{\log N}{\log \log N} \right)^{\frac{1}{4}} \min_{\gamma \in \Omega} | \omega(\gamma) | \tag{8.10}$$

provided $N = | \Omega |$ is sufficiently large.

So, if we can arrange for $\Omega = \Omega_j$ to vary in such a way that the right-hand side of (8.10) tends to infinity with $j$, the substance of 7.6 will lead to a continuous $f$ satisfying $\mathrm{sp}(f) \subseteq \Gamma_0$ and

$$\overline{\lim_{j \to \infty}} \operatorname{Re} S_{\Delta_j}^\omega f(0) = \infty. \tag{8.11}$$

Taking the most familiar case, in which $G = T$, $\Gamma = Z$ and $\Phi = \{0\}$, and supposing $\Delta = \Omega$ to range over a sequence $(\Delta_j)$ of finite subsets of $Z$ such that, if $N_j = | \Delta_j |$,

$$\lim_j \left( \frac{\log N_j}{\log \log N_j} \right)^{\frac{1}{4}} \min_{n \in \Delta_j} | \omega(n) | = \infty,$$

the construction will lead to a continuous $f$ on $T$ such that

$$\overline{\lim_j} \operatorname{Re} S_{\Delta_j}^\omega f(0) = \infty.$$

In particular, taking $\Delta_j = \{n \in Z : 2^j \leqq n < 2^{j+1}\}$ it can be arranged that

$$\sum_{n \in Z} \frac{\pm \hat{f}(n)}{(\log(2+|n|))^{\alpha}}$$

diverges for any preassigned distribution of signs $\pm$ and any preassigned $\alpha < \frac{1}{4}$.

Of course, much stronger results are derivable by using random (and unspecifiable!) changes of sign, but there seems little hope of making this even remotely constructive.

## § 9. *Discussion of case* (ii) : G 0-*dimensional*

9.1 In this case there is ([7], (7.7)) a base of neighbourhoods of zero in $G$ formed of compact open subgroups $W$. For each such $W$ the annihilator $\Delta = W^{\circ}$ in $\Gamma$ of $W$ is a finite subgroup of $\Gamma$. Define

$$k_W = \lambda_G(W)^{-1} \times \text{characteristic function of } W. \tag{9.1}$$

Then $k_W$ is continuous, $k_W \geqq 0$, $\int_G k_W \, d\lambda_G = 1$. The transform $\hat{k}_W$ of $k_W$ is plainly equal to unity on $\Delta$. On the other hand, since $W$ is a subgroup, we have for $a \in W$ and $\gamma \in \Gamma$

$$\begin{aligned} \hat{k}_W(\gamma) &= \int_G k_W(x) \overline{\gamma(x)} \, d\lambda_G(x) = \int_G k_W(x+a) \overline{\gamma(x)} \, d\lambda_G(x) \\ &= \int_G k_W(y) \overline{\gamma(y-a)} \, d\lambda_G(y) \\ &= \gamma(a) \hat{k}_W(\gamma), \end{aligned}$$

which shows that $\hat{k}_W(\gamma) = 0$ if $\gamma \in \Gamma \setminus \Delta$. Thus $\hat{k}_W$ is the characteristic function of $\Delta$, and so

$$k_W = D_{W^{\circ}}. \tag{9.2}$$

By (9.1) and (9.2), a routine argument shows that, if $1 \leqq p < \infty$ and $f \in L^p(G)$, then

$$f = \lim_W S_{W^{\circ}} f \tag{9.3}$$

in $L^p(G)$; and that (9.3) holds uniformly for any continuous $f$.

9.2 Proof of 7.4 (ii). If $\Gamma_0$ is any countably infinite subgroup of $\Gamma$ we can choose a sequence $W_j$ of compact open subgroups of $G$ such that

$W_{j+1} \subseteq W_j$ and $\Gamma_0 \subseteq \bigcup_{j=1}^{\infty} W_j^\circ$, where $W_j^\circ$ is a finite subgroup of $\Gamma$ and $W_j^\circ \subseteq W_{j+1}^\circ$. The $\Delta_j = W_j^\circ \cap \Gamma_0$ satisfy (7.2) and, from (9.3),

$$f = \lim_j S_{\Delta_j} f \tag{9.4}$$

uniformly for any continuous $f$ with $\mathrm{sp}(f) \subseteq \Gamma_0$. This verifies the statements made in 7.4 (ii).

9.3 By using the results in [3], more can be said in case (ii) of 7.4; cf. [3], Theorem (2.9) and Example (4.8).

Let $f \in L^1(G)$ and let $\Gamma_0$ be any countable subgroup of $\Gamma$ containing $\mathrm{sp}(f)$. Choose the $W_j$ as in 9.2. Then, apart from the fact that $(W_j)$ is not in general a base at 0 in $G$ (they can be chosen to be so if and only if $G$ is first countable), $(W_j)$ is an open-compact $D''$-sequence ([3], p. 188). The proof of Theorem (2.5) of [3] is easily modified to show that

$$f(x) = \lim_{j \to \infty} S_{W_j^\circ} f(x) \tag{9.5}$$

holds for almost all $x \in G$. Moreover, Theorem (2.7) of [3] applies to show that the majorant function

$$S^* f(x) = \sup_{j \in N} \left| S_{W_j^\circ} f(x) \right| \tag{9.6}$$

satisfies the estimates

$$\| S^* f \|_p \leqq 2 (p (p-1)^{-1})^{\frac{1}{p}} \| f \|_p \quad (1 < p < \infty) \tag{9.7}$$

$$\| S^* f \|_1 \leqq 2 + 2 \int_G |f| \log^+ |f| \, d\lambda_G, \tag{9.8}$$

$$\| S^* f \|_p \leqq 2 (1-p)^{\frac{1}{p}} \| f \|_1 \quad (0 < p < 1). \tag{9.9}$$

In particular, the convergence in (9.5) is dominated whenever

$$|f| \log^+ |f| \in L^1(G).$$

A more immediate consequence of (9.1) and (9.2) is a strong version of localisability of the convergence of Fourier series: if $f \in L^1(G)$ vanishes a.e. on some neighbourhood of $x_0 \in G$, we can choose the $W_j$ so that $S_{\Delta_j} f(x_0) = 0$ for every sufficiently large $j$. [A suitable choice of $W_j$ may be made once for all, independent of $f$, if $G$ is first countable.] Nothing similar is true for general $G$; see, for example, [11], Vol. II, pp. 304-305.

## § 10. *Concerning the polynomials* $Q_j$.

There is no difficulty in making fairly explicit the construction of t.p.s $Q_j$ of the type employed in 7.6.

For $p > 0$, $t \geqq 0$ define

$$h_p(t) = \begin{cases} 1 & \text{if } t \leqq p, \\ 2\left(1 - \dfrac{t}{2p}\right) & \text{if } p \leqq t \leqq 2p, \\ 0 & \text{if } t \geqq 2p. \end{cases} \tag{10.1}$$

For all complex $z$ define

$$f_p(z) = \begin{cases} 0 & \text{if } z = 0, \\ |z|^{-1}\,\bar{z}\,h_p(|z|) & \text{if } z \neq 0. \end{cases} \tag{10.2}$$

Write

$$\left.\begin{aligned} E_n(z) &= \pi^{-1}\, n \exp(-n\,|z|^2, \\ P_{n,k}(z) &= \pi^{-1}\, n \sum_{j=0}^{k} \frac{(-1)^j}{j!} (n|z|^2)^j \end{aligned}\right\} \tag{10.3}$$

Let $\mu$ denote Lebesgue measure on $C$ (identified with $R^2$ in the canonical fashion).

It is then routine to verify that

$$\left.\begin{aligned} &\| E_n * f_p \|_\infty \leqq \| f_p \|_\infty = 1, \\ &\lim_{n\to\infty} E_n * f_p = f_p \end{aligned}\right\} \tag{10.4}$$

uniformly on any compact set omitting 0. From this it follows that to every $p > 0$ and every positive integer $\nu$ correspond positive integers $\bar{n}(p, \nu)$, $\bar{k}(p, \nu)$ such that

$$\left.\begin{aligned} &\left| |z|^{-1}\bar{z} - f_p * P_{\bar{n},\bar{k}}(z) \right| \leqq \frac{1}{\nu} \text{ for } \frac{1}{\nu} \leqq |z| \leqq p, \\ &\left| f_p * P_{\bar{n},\bar{k}}(z) \right| \leqq 1 + \frac{1}{\nu} \text{ for } |z| \leqq p. \end{aligned}\right\} \tag{10.5}$$

Now

$$f_p * P_{\bar{n},\bar{k}}(z) = q_{p,\nu}(z, \bar{z}), \tag{10.6}$$

where

$$q_{p,\nu}(X, Y) = \pi^{-1}\,\bar{n}(p, \nu) \sum_{j=0}^{\bar{k}(p,\nu)} \frac{(-\bar{n}(p, \nu))^j}{j!} \sum_{l=0}^{j} \sum_{m=0}^{j} \binom{j}{l}\binom{j}{m} X^l\, Y^m$$

$$(-1)^{l+m} \int \zeta^{j-l}\bar{\zeta}^{j-m} f_p(\zeta)\, d\mu(\zeta)$$

$$= \sum_{l,m=0}^{\bar{k}(p,\nu)} C_{p,\nu}(l, m)\, X^l\, Y^m. \tag{10.7}$$

It is easily verifiable that the $C_{p,\nu}(l, m)$ are real-valued.

If $\theta$ is a bounded measurable function on $G$ and

$$\overset{\circ}{Q}_{p,\nu} = q_{p,\nu}(\theta, \bar{\theta}),\ p \geqq \|\theta\|_\infty, \tag{10.8}$$

we have from (10.5)

$$\left.\begin{aligned} &\left|\, |\theta|^{-1}\bar{\theta} - \overset{\circ}{Q}_{p,\nu} \right| \leqslant \frac{1}{\nu} \text{ whenever } |\theta| \geqq \frac{1}{\nu}, \\ &|\overset{\circ}{Q}_{p,\nu}| \leqq 1 + \frac{1}{\nu} \text{ everywhere on } G. \end{aligned}\right\} \tag{10.9}$$

If $\theta$ is a t.p., then $\overset{\circ}{Q}_{p,\nu}$ is a t.p. and

$$\mathrm{sp}\,(\overset{\circ}{Q}_{p,\nu}) \subseteq [\mathrm{sp}\,(\theta)]. \tag{10.10}$$

From (10.9) we obtain

$$\left|\, |\theta| - \theta\, \overset{\circ}{Q}_{p,\nu} \right| \leqq \begin{cases} \nu^{-1}\,|\theta| \text{ whenever } |\theta| \geqslant \dfrac{1}{\nu}, \\ \left(2+\dfrac{1}{\nu}\right)|\theta| \text{ everywhere,} \end{cases}$$

whence it follows that, if $\theta \neq 0$,

$$\left|\int_G \theta\, \overset{\circ}{Q}_{p,\nu}\, d\lambda G\right| \geqq (1-\nu^{-1})\,\|\theta\|_1 - \nu^{-1}(2+\nu^{-1})$$

$$\geqq (1-2\nu^{-\frac{1}{2}})\,\|\theta\|_1 \tag{10.11}$$

provided $\nu \geqq 9\,\|\theta\|_1^{-2}$.

Taking $\theta = D_{\Delta_j}$ and $p_j \geqq \|D_{\Delta_j}\|$, the trigonometric polynomials

$$Q_j' = \left(1+\frac{1}{\nu}\right)^{-1} \overset{\circ}{Q}_{p_j,\nu} = \left(1+\frac{1}{\nu}\right)^{-1} q_{p_j,\nu}(D_{\Delta_j}, \bar{D}_{\Delta_j}) \tag{10.12}$$

are then seen from (10.9), (10.10) and (10.11) to satisfy

$$\left.\begin{aligned} \| Q_j' \| &\leqq 1, \\ \mathrm{sp}\,(Q_j') &\subseteq [\Delta_j], \\ \left| \int v\, D_{\Delta_j}\, Q_j'\, d\lambda_G \right| &\geqq (1-3v^{-\frac{1}{2}}) \| D_{\Delta_j} \|_1 \end{aligned}\right\} \tag{10.13}$$

provided $v$ is chosen $\geqq 9 \| D_{\Delta_j} \|_1^{-1}$. In view of (7.6), we may choose the integer $v \geqq \max_j (36, 9 \| D_{\Delta_j} \|_1^{-1})$. Then (10.13) shows that there are unimodular complex numbers $\xi_j$ such that the $Q_j = \xi_j Q_j'$ satisfy (7.7).

## Appendix

### *Rudin-Shapiro sequences*

A.1 Notations and definitions. As hitherto, all topological groups $G$ are assumed to be Hausdorff; and, for any locally compact group $G$, $\lambda_G$ will denote a selected left Haar measure, with respect to which the Lebesgue spaces $L^p(G)$ are to be formed. $C_c(G)$ denotes the set of complex-valued continuous functions on $G$ having compact supports.

If $X$ and $Y$ are topological groups, Hom $(X, Y)$ denotes the set of continuous homomorphisms of $X$ into $Y$.

Suppose henceforth $G$ to be locally compact. As in 5.1, if $k \in C_c(G)$, $T_k$ will denote the convolution operator

$$f \mapsto f * k$$

with domain $C_c(G)$ and range in $C_c(G)$; and $\| k \|_{p,q}$ will denote the $(p, q)$-norm of this operator, i.e., the smallest real number $m \geqq 0$ such that

$$\| f * k \|_q \leqq m \| f \|_p \quad (f \in C_c(G)).$$

It is well-known that, if $G$ is Abelian, $\| k \|_{2,2}$ is equal to

$$\| \hat{k} \|_\infty = \sup_{\gamma \in \Gamma} | \hat{k}(\gamma) |,$$

where $\Gamma$ is the character group of $G$ and $\hat{k}$ is the Fourier transform of $k$. (Something similar is true whenever $G$ is compact, but we shall not use this.)

$U$-RS-sequences on $G$ are as defined in 5.4.