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satisfying  $1 \leq p < 2 < q \leq \infty$ , the series (6.6) converges normally in  $L_p^q(G)$  to  $T$ . Next,  $T$  is the limit in  $E$  of

$$S_r = \sum_{n=1}^r \omega_n T_{K_n}$$

as  $r \rightarrow \infty$  and, since it is plain that  $\text{supp } S_r \subseteq \Omega$  for every  $r$ , (ii) is easily derived. Finally, if  $\hat{T}$  were a measure  $\mu$ , it would necessarily be the case that  $\text{supp } \mu \subseteq \bar{\Omega}$  and so, for every  $n \in N$ , one would have by (6.1) and (6.4)

$$\begin{aligned} f_n(T) &= |u_n * Tv_n(0)| = \left| \int_{\Gamma} \hat{u}_n \hat{v}_n d\mu \right| \\ &\leq |\mu|(\bar{\Omega}), \end{aligned}$$

which is finite since  $\Omega$  is relatively compact. However, this plainly would entail  $f^*(T) < \infty$ , in conflict with (6.8), so that  $T$  cannot be a measure and (iii) is verified. This completes the proof.

6.4 REMARK. Theorem 6.3 was proved by Hörmander ([14], Theorem 1.9) for  $G = R^n$  and any given pair  $(p, q)$  satisfying  $1 \leq p < 2 < q \leq \infty$ , this result being extended to a general noncompact LCA  $G$  by Gaudry [5]. The argument given by Hörmander (loc. cit. Theorem 1.6 and the remark immediately following) for the case  $G = R^n$  can also be extended to a general LCA  $G$  and shows that, if either  $q \leq 2$  or  $p \geq 2$ , then every  $T \in L_p^q(G)$  is such that  $\hat{T}$  is a measure [and indeed a measure of the form  $\psi \lambda_{\Gamma}$ , where  $\psi \in L_{loc}^2(\Gamma)$  if  $q \leq 2$  and  $\psi \in L_{loc}^p(\Gamma)$  if  $p \geq 2$ , and so  $\psi \in L_{loc}^2(\Gamma)$  in either case]. Thus the hypotheses made in Theorem 6.3 about  $p$  and  $q$  are necessary for the validity of the conclusion.

### PART 3: APPLICATIONS TO FOURIER SERIES

#### § 7. Applications to divergence of Fourier series.

7.1 Throughout §§ 7-10,  $G$  will denote an infinite Hausdorff compact Abelian group with character group  $\Gamma$ , and  $\lambda_G$  the Haar measure on  $G$ , normalised so that  $\lambda_G(G) = 1$ . For any  $f \in L^1(G)$ ,  $\hat{f}$  will denote the Fourier transform of  $f$ ; for any finite subset  $\Delta$  of  $\Gamma$ ,

$$S_{\Delta} f = \sum_{\gamma \in \Delta} \hat{f}(\gamma) \gamma \tag{7.1}$$

is the  $\Delta$ -partial sum of the Fourier series of  $f$ ; and  $\text{sp}(f)$  will stand for

the spectrum of  $f$ , i.e., for the support  $\text{supp } \hat{f} = \{\gamma \in \Gamma : \hat{f}(\gamma) \neq 0\}$  of  $\hat{f}$ . The term “trigonometric polynomial” will frequently be abbreviated to “t.p.”. In addition,  $\Phi$  will denote the largest torsion subgroup of  $\Gamma$  ([7], (A.4)), and  $\pi$  the natural map of  $\Gamma$  onto  $\Gamma/\Phi$ . If  $\Delta$  denotes a subset of  $\Gamma$ ,  $[\Delta]$  will stand for the subgroup of  $\Gamma$  generated by  $\Delta$ .

By a (*convergence*) *grouping* we shall mean a sequence  $\mathcal{D} = (\Delta_j)_{j \in N} = (\Delta_j)$  of finite subsets  $\Delta_j$  of  $\Gamma$  such that

$$\left. \begin{aligned} \Delta_j &\subseteq \Delta_{j+1} \quad (j \in N); \\ \bigcup_{j=1}^{\infty} \Delta_j = \Gamma_0 &\text{ is a subgroup of } \Gamma, \text{ said to be} \\ &\text{covered by } \mathcal{D}; \\ \text{for each } j \in N, \Delta_j &= \Omega_j + \Lambda_j, \text{ where } \Lambda_j \text{ is a} \\ &\text{nonvoid finite subset of } \Phi \text{ and } \Omega_j \text{ is a finite} \\ &\text{subset of } \Gamma \text{ such that } \pi|_{\Omega_j} \text{ is 1-1.} \end{aligned} \right\} \quad (7.2)$$

[The first two conditions are natural enough in the context described in 7.3, but the third is less so and may well be pointless.] The grouping  $\mathcal{D}$  is said to be of *infinite type* if and only if  $\pi(\Gamma_0)$  is infinite.

7.2 EXAMPLES. (i) Let  $\Gamma_0$  be any countable subgroup of  $\Gamma$  such that  $\Gamma_0 \cap \Phi = \{0\}$ ; for example,  $\Gamma_0 = \{n\gamma_0 : n \in \mathbb{Z}\}$ , where  $\gamma_0 \in \Gamma \setminus \Phi$ . Then a grouping  $\mathcal{D}$  covering  $\Gamma_0$  results whenever  $\Lambda_j = \{0\}$  and  $\Delta_j = \Omega_j$  for every  $j \in N$ , where  $(\Omega_j)_{j \in N}$  is any increasing sequence of finite subsets of  $\Gamma_0$  with union equal to  $\Gamma_0$ . This grouping is of infinite type if and only if  $\Gamma_0$  is infinite.

(ii) If  $G$  is connected, and if  $\Gamma_0$  is any countable subgroup of  $\Gamma$ , then ([10], 2.5.6 (c), 8.1.2 (a) and (b) and 8.1.6)  $\Gamma_0$  is an ordered group isomorphic to a discrete subgroup of  $R$ . Assuming  $\Gamma_0 \neq \{0\}$ ,  $\Gamma_0$  has a smallest positive element  $\gamma_0$  and  $\Gamma_0 = \{n\gamma_0 : n \in \mathbb{Z}\}$ . A natural grouping  $\mathcal{D}$  covering  $\Gamma_0$  is that in which  $\Lambda_j = \{0\}$  and

$$\Delta_j = \Omega_j = \{n\gamma_0 : n \in \mathbb{Z}, |n| \leq j\}$$

for every  $j \in N$ ; this grouping is of infinite type.

7.3 A grouping  $\mathcal{D} = (\Delta_j)_{j \in N}$  will be thought of as specifying one of the many possible ways in which one may interpret the convergence of Fourier series of functions  $f$  on  $G$  satisfying  $sp(f) \subseteq \Gamma_0$ , namely, as convergence of the corresponding sequence of partial sums  $(S_{\Delta_j} f)_{j \in N}$ .

Indeed, the conditions (7.2) guarantee that  $\lim_{j \rightarrow \infty} S_{\Delta_j} f = f$  for all sufficiently regular such functions  $f$ . However, our concern rests with the possibility of constructing continuous functions  $f$  on  $G$  satisfying

$$\text{sp}(f) \subseteq \Gamma_0, \quad \overline{\lim_{j \rightarrow \infty} \text{Re } S_{\Delta_j} f(0)} = \infty. \quad (7.3)$$

It will appear that the possibilities exhibit a fairly clear dichotomy, depending largely upon whether  $G$  is or is not 0-dimensional.

In the first place, it will emerge in 7.6 that the construction principle of § 2, applied to the Banach space  $E = C(G)$  of continuous complex valued functions on  $G$  [with norm  $\|\cdot\|$  equal to the maximum modulus] and to sequences of gauges of the type

$$f \mapsto \text{Re } S_{\Delta} f(0) = \text{Re} \int_G D_{\Delta} f d\lambda_G, \quad (7.4)$$

where  $D_{\Delta}$  stands for the “Dirichlet function”

$$D_{\Delta} = \sum_{\gamma \in \Delta} \bar{\gamma}, \quad (7.5)$$

shows that the problem hinges on the existence of groupings  $\mathscr{D}$  for which

$$\rho_j = \|D_{\Delta_j}\|_1 = \int_G |D_{\Delta_j}| d\lambda_G \rightarrow \infty. \quad (7.6)$$

Accordingly, and in view of the fact ([7], (24.26)) that  $G$  is 0-dimensional if and only if  $\Gamma$  coincides with  $\Phi$ , it emerges that the dichotomy referred to may be expressed in the following way.

7.4 Two cases arise, namely:

(i)  $G$  is not 0-dimensional (i.e.,  $\Phi \neq \Gamma$ ). Then (see Example 7.2 (i)) there exist groupings  $\mathscr{D} = (\Delta_j)$  of infinite type; and, for any such grouping, one can construct (fairly explicitly, as described in 7.6) continuous functions  $f$  on  $G$  satisfying (7.3). In particular [cf. Example 7.2 (i)], if  $\Gamma_0$  is any countably infinite subgroup of  $\Gamma$  satisfying  $\Gamma_0 \cap \Phi = \{0\}$ , and if  $(\Delta_j)_{j \in \mathbb{N}}$  is any increasing sequence of finite subsets of  $\Gamma_0$  with union  $\Gamma_0$ , we can construct a continuous  $f$  on  $G$  satisfying (7.3).

(ii)  $G$  is 0-dimensional (i.e.,  $\Phi = \Gamma$ ). Then there exists no grouping of infinite type. However, given any countable subgroup  $\Gamma_0$  of  $\Gamma$ , there are groupings  $\mathscr{D} = (\Delta_j)$  covering  $\Gamma_0$ , in which  $\Omega_j = \{0\}$  and  $\Delta_j = \Delta_j$  is a finite subgroup of  $\Gamma_0$ , and for which

$$f = \lim_{j \rightarrow \infty} S_{\Delta_j} f$$



uniformly on  $G$  for every continuous  $f$  satisfying  $\text{sp}(f) \subseteq \Gamma_0$ .

Case (i) will be dealt with in § 8, case (ii) in § 9. The groupings described in case (ii) prove to be exceptional in various ways; see 9.3.

**7.5 REMARK.** Perhaps it should be stressed here that, if  $\Gamma_0$  is any infinite subgroup of  $\Gamma$ , there is no obstacle to constructing continuous functions  $f$  such that  $\text{sp}(f) \subseteq \Gamma_0$  and finite subsets  $\Delta_j \subseteq \Delta_{j+1}$  of  $\Gamma_0$  for which

$$\lim_j S_{\Delta_j} f(0) = \infty.$$

[One has in fact only to construct a continuous  $f$  such that  $\text{sp}(f) \subseteq \Gamma_0$  and  $\sum_{\gamma \in \Gamma} |\hat{f}(\gamma)| = \infty$ ; it is then trivial that there exist finite subsets  $\Delta$  of  $\Gamma_0$  for which  $|S_{\Delta} f(0)|$  is arbitrarily large, so that we can choose a sequence  $(\Delta_j)$  for which  $\Delta_j \subseteq \Delta_{j+1}$  and  $|S_{\Delta_j} f(0)| \rightarrow \infty$  with  $j$ .] However, the sets  $\Delta_j$  obtained this way will not [and, in view of 7.4 (ii), cannot] in general be such that  $\bigcup_{j=1}^{\infty} \Delta_j = \Gamma_0$ . For more details, see A.5.1 and A.5.2 of the Appendix.

**7.6** Suppose one is given a grouping  $\mathcal{D} = (\Delta_j)_{j \in \mathbb{N}}$  covering  $\Gamma_0$  and satisfying (7.6). As is described in § 10, one may construct polynomials  $q_{p_j, \nu}$  in two indeterminates over the real field ( $\nu$  being a suitable fixed integer not less than 36 and  $p_j$  any positive number not less than  $\|D_{\Delta_j}\|_{\infty}$ ) such that, for suitable unimodular complex numbers  $\xi_j$ , the t.p.s

$$Q_j = \xi_j \left(1 + \frac{1}{\nu}\right)^{-1} q_{p_j, \nu}(D_{\Delta_j}, \bar{D}_{\Delta_j})$$

satisfy

$$\left. \begin{aligned} \|Q_j\| &\leq 1, \text{sp}(Q_j) \subseteq [\Delta_j] \subseteq \Gamma_0, \\ S_{\Delta_j} Q_j(0) &= \int_G D_{\Delta_j} Q_j d\lambda_G \text{ is real and } \geq \frac{1}{2} \rho_j. \end{aligned} \right\} \quad (7.7)$$

In view of (7.2), (7.6) and (7.7), one may choose inductively a sequence  $(j_n)_{n \in \mathbb{N}}$  of positive integers so that

$$\left. \begin{aligned} S_{\Delta_{j_n}} Q_{j_n}(0) &\text{ is real and } > n^3, \\ j_n &< j_{n+1}, \text{sp}(Q_{j_n}) \subseteq \Gamma_0. \end{aligned} \right\} \quad (7.8)$$

Accordingly, the t.p.s

$$u_n = n^{-2} Q_{j_n}$$

satisfy the conditions

$$\left. \begin{aligned} \text{sp}(u_n) &\subseteq \Gamma_0, \sum_{n=1}^{\infty} \|u_n\| < \infty \\ S_{\Delta_{j_n}} u_n(0) &\text{ is real and } > n. \end{aligned} \right\} \quad (7.9)$$

At this point the construction in § 2 will yield integers  $0 < n_1 < n_2 < \dots$  and specifiable sequences  $(\gamma_p)_{p \in \mathbb{N}}$  of positive numbers such that each function of the form

$$f = \sum_{p=1}^{\infty} \gamma_p u_{n_p}$$

is continuous and satisfies

$$\text{sp}(f) \subseteq \Gamma_0, \lim_{p \rightarrow \infty} \text{Re } S_{\Delta_{j_{n_p}}} f(0) = \infty. \quad (7.10)$$

A fortiori,  $f$  satisfies (7.3).

We add here that, if the  $\Delta_j$  are symmetric, the  $D_{\Delta_j}$  are real-valued, and we may work throughout with real-valued functions, replacing  $\text{Re } S_{\Delta_j} f$  by  $S_{\Delta_j} f$  everywhere.

## § 8. Discussion of case (i) : $G$ not 0-dimensional

8.1 In this case  $\Phi \neq \Gamma$ , and we begin by considering a finite subset of  $\Gamma$  of the form

$$\Delta = \Omega + \Lambda, \quad (8.1)$$

where  $\Omega$  and  $\Lambda$  are finite subsets of  $\Gamma$  such that  $\pi|_{\Omega}$  is 1-1 and  $\emptyset \neq \Lambda \subseteq \Phi$ . We aim to show that (for a suitable absolute constant  $k > 0$ )

$$\|D_{\Delta}\|_1 \geq k \left( \frac{\log N}{\log \log N} \right)^{\frac{1}{4}}, \quad (8.2)$$

provided  $N = |\Omega|$  (the cardinal number of  $\Omega$ ) is sufficiently large.

8.2 PROOF OF (8.2). Introduce  $H$  as the annihilator in  $G$  of  $\Phi$  and identify in the usual way the dual of  $H$  with  $\Gamma/\Phi$ . Likewise identify the dual of  $K = G/H$  with  $\Phi$  ([7], (24.11)).

We then have

$$\begin{aligned} \|D_A\|_1 &= \int_G \left| \sum_{\gamma \in A} \gamma \right| d\lambda_G \\ &= \int_{G/H} d\lambda_{G/H}(\bar{x}) \int_H \left| \sum_{\theta \in \Omega} \sum_{\phi \in A} \theta(x+y) \phi(x+y) \right| d\lambda_H(y), \end{aligned}$$

the inner integral being viewed as a function of  $\bar{x} = x+H$ . Thus, writing  $\bar{\theta}$  for  $\pi(\theta)$  and noting that  $\phi(y) = 1$  for  $\phi \in A \subseteq \Phi$  and  $y \in H$ , we obtain

$$\|D_A\|_1 = \int_{G/H} d\lambda_{G/H}(\bar{x}) \int_H \left| \sum_{\theta \in \Omega} \alpha(\theta, x) \bar{\theta}(y) \right| d\lambda_H(y), \quad (8.3)$$

where

$$\alpha(\theta, x) = \theta(x) \sum_{\phi \in A} \phi(x).$$

Now, since the dual of  $H$  (namely  $\Gamma/\Phi$ ) is torsion-free ([7], (A.4)), Theorem A of [8] shows that (for a suitable absolute constant  $k > 0$ ) we have

$$\begin{aligned} \int_H \left| \sum_{\theta \in \Omega} \alpha(\theta, x) \bar{\theta}(y) \right| d\lambda_H(y) &\geq k \left( \frac{\log N}{\log \log N} \right)^{\frac{1}{4}} \min_{\theta \in \Omega} |\alpha(\theta, x)| \\ &= k \left( \frac{\log N}{\log \log N} \right)^{\frac{1}{4}} \left| \sum_{\phi \in A} \phi(\bar{x}) \right|, \end{aligned} \quad (8.4)$$

since  $|\theta(x)| = 1$  and  $\phi(x)$  depends only  $\bar{x}$ . By (8.3) and (8.4),

$$\|D_A\|_1 \geq k \left( \frac{\log N}{\log \log N} \right)^{\frac{1}{4}} \int_{G/H} \left| \sum_{\phi \in A} \phi(\bar{x}) \right| d\lambda_{G/H}(\bar{x}). \quad (8.5)$$

Since  $A \neq \emptyset$ , the remaining integral is not less than the maximum modulus of the Fourier transform of the function  $\bar{x} \mapsto \sum_{\phi \in A} \phi(\bar{x})$ , i.e., is not less than unity. Thus, (8.2) follows from (8.5).

**8.3 PROOF OF 7.4 (i).** The conclusions stated in case (i) of 7.4 are now almost immediate. If  $\mathcal{D} = (A_j)_{j \in N}$  is a grouping of infinite type covering  $\Gamma_0$ ,  $|\pi(A_j)| \rightarrow \infty$  and so, since  $A_j \subseteq \Phi$ ,  $|\pi(\Omega_j)| \rightarrow \infty$ . Then (8.2) shows that (7.6) is satisfied, and it remains only to refer to 7.6.

**8.4 SUPPLEMENTARY REMARKS.** The fact that, when  $G$  is not 0-dimensional, (7.6) holds for suitable subgroups  $\Gamma_0$  of  $\Gamma$  and suitable groupings  $\mathcal{D} = (A_j)_{j \in N}$  covering  $\Gamma_0$  can be derived without appeal to Theorem A

of [8]. To do this, it suffices to take  $\gamma_k \in \Gamma \setminus \Phi$  ( $k = 1, 2, \dots, m$ ) such that the family  $(\gamma_k)_{1 \leq k \leq m}$  is independent (see [7], (A.10)), define

$$\Gamma_0 = \left\{ \sum_{k=1}^m n_k \gamma_k : n_k \in \mathbb{Z} \text{ for } k = 1, 2, \dots, m \right\},$$

and make use of the formula

$$\begin{aligned} \int_G F(\gamma_1(x), \dots, \gamma_m(x)) d\gamma_G(x) \\ = (2\pi)^{-m} \int_0^{2\pi} \dots \int_0^{2\pi} F(e^{it_1}, \dots, e^{it_m}) dt_1 \dots dt_m, \end{aligned} \quad (8.6)$$

valid for every  $F \in C(T^m)$ , where  $T$  denotes the circle group. (Recall that  $\sum_{k=1}^m n_k \gamma_k$  denotes the character  $x \mapsto \gamma_1(x)^{n_1} \dots \gamma_m(x)^{n_m}$  of  $G$ .) It then appears that (7.6) holds when one takes

$$\Delta_j = \left\{ \sum_{k=1}^m n_k \gamma_k : |n_k| \leq r_{j,k} \text{ for } k = 1, 2, \dots, m \right\},$$

where the  $r_{j,k}$  are positive integers satisfying  $r_{j,k} \leq r_{j,k+1}$  and  $\lim_{j \rightarrow \infty} r_{j,k} = \infty$ . Moreover, when  $m = 1$ , the Cohen-Davenport result (essentially Theorem A of [8] for the case  $G = T$ ) shows that (7.6) holds for every grouping  $\mathcal{D}$  covering  $\Gamma_0$ .

The verification of (8.6) is simple. First note that, if  $G$  and  $G'$  are compact groups, and if  $\phi$  is a continuous homomorphism of  $G$  into  $G'$ , then

$$\int_G (F \circ \phi) d\lambda_G = \int F d\lambda_{\phi(G)} \quad (8.7)$$

for every  $F \in C(G')$ . (This is a consequence of the fact that  $F \mapsto \int_G (F \circ \phi) d\lambda_G$  is invariant under translation by elements of  $\phi(G)$ , combined with the uniqueness of the normalised Haar measure on a compact group.) Taking  $G' = T^m$  and  $\phi : x \mapsto (\gamma_1(x), \dots, \gamma_m(x))$ , the stated conditions on the  $\gamma_k$  are just adequate to ensure that the annihilator in  $\mathbb{Z}^m$  (identified in the canonical fashion with the dual of  $T^m$ ) of  $\phi(G)$  is  $\{(0, \dots, 0)\}$  and so ([7], (24.10)) that  $\phi(G) = T^m$ . Accordingly, (8.6) appears as a special case of (8.7).

It is perhaps worth indicating that special cases of (8.7) can be exploited in other ways. For example, suppose more generally that  $\kappa$  is an arbitrary nonvoid set and that  $(\gamma_k)_{k \in \kappa}$  is a finite or infinite independent family of elements of  $\Gamma \setminus \Phi$ . Denote by  $\Gamma_0$  the subgroup of  $\Gamma$  generated by  $\{\gamma_k : k \in \kappa\}$ . Taking  $G' = T^\kappa$  and  $\phi : x \mapsto (\gamma_k(x))_{k \in \kappa}$ , one may use (8.7) in a similar fashion to show that there is an isometric isomorphism  $F \leftrightarrow F \circ \phi = f$  between  $L^p(T^\kappa)$  (or  $C(T^\kappa)$ ) and the subspace of  $L^p(G)$  (or  $C(G)$ ) formed of those  $f \in L^p(G)$  or  $C(G)$  such that  $\text{sp}(f) \subseteq \Gamma_0$ . Moreover, if one identifies in the canonical fashion the dual of  $T^\kappa$  with the weak

direct product  $Z^{\kappa*}$ , the said isomorphism is such that  $\hat{F} = \hat{f} \circ \phi'$ , where  $\phi'$  is the isomorphism of  $Z^{\kappa*}$  onto  $\Gamma_0$  defined by  $(n_k) \rightarrow \sum_{k \in \kappa} n_k \gamma_k$ .

One consequence of this may be expressed roughly as follows: If the compact Abelian group  $G$  is such that  $\Gamma \setminus \Phi$  contains an independent family of (finite or infinite) cardinality  $m$ , then Fourier series on  $G$  behave, in respect of convergence or summability, no better than do Fourier series on  $T^m$ .

Another consequence is that, if  $\Delta$  is a subset of  $\Gamma_0$ , then  $\Delta$  is a Sidon (or  $\Lambda(p)$ ) subset of  $\Gamma$  if and only if  $\phi'^{-1}(\Delta)$  is a Sidon (or  $\Lambda(p)$ ) subset of  $Z^{\kappa*}$ .

8.5 FURTHER RESULTS. Theorem A of [8] implies something stronger than (8.2), namely: if  $\omega$  is any complex-valued function on  $\Gamma$  such that

$$\omega(\gamma + \phi) = \omega(\gamma) \quad (\gamma \in \Gamma, \phi \in \Phi), \quad (8.8)$$

so that  $\omega$  can be regarded as a function on  $\Gamma/\Phi$ , and if we write

$$D_{\Delta}^{\omega} = \sum_{\gamma \in \Delta} \omega(\gamma) \bar{\gamma}, \quad S_{\Delta}^{\omega} f = \sum_{\gamma \in \Delta} \omega(\gamma) \hat{f}(\gamma), \quad (8.9)$$

then, for  $\Delta = \Omega + \Lambda$  as in (8.1), we have

$$\|D_{\Delta}^{\omega}\|_1 \geq k \left( \frac{\log N}{\log \log N} \right)^{\frac{1}{4}} \min_{\gamma \in \Omega} |\omega(\gamma)| \quad (8.10)$$

provided  $N = |\Omega|$  is sufficiently large.

So, if we can arrange for  $\Omega = \Omega_j$  to vary in such a way that the right-hand side of (8.10) tends to infinity with  $j$ , the substance of 7.6 will lead to a continuous  $f$  satisfying  $\text{sp}(f) \subseteq \Gamma_0$  and

$$\overline{\lim}_{j \rightarrow \infty} \text{Re } S_{\Delta_j}^{\omega} f(0) = \infty. \quad (8.11)$$

Taking the most familiar case, in which  $G = T$ ,  $\Gamma = Z$  and  $\Phi = \{0\}$ , and supposing  $\Delta = \Omega$  to range over a sequence  $(\Delta_j)$  of finite subsets of  $Z$  such that, if  $N_j = |\Delta_j|$ ,

$$\lim_j \left( \frac{\log N_j}{\log \log N_j} \right)^{\frac{1}{4}} \min_{n \in \Delta_j} |\omega(n)| = \infty,$$

the construction will lead to a continuous  $f$  on  $T$  such that

$$\overline{\lim}_j \text{Re } S_{\Delta_j}^{\omega} f(0) = \infty.$$

In particular, taking  $\Delta_j = \{n \in \mathbb{Z} : 2^j \leq n < 2^{j+1}\}$  it can be arranged that

$$\sum_{n \in \mathbb{Z}} \frac{\pm \hat{f}(n)}{(\log(2 + |n|))^\alpha}$$

diverges for any preassigned distribution of signs  $\pm$  and any preassigned  $\alpha < \frac{1}{4}$ .

Of course, much stronger results are derivable by using random (and unspecifiable!) changes of sign, but there seems little hope of making this even remotely constructive.

### § 9. Discussion of case (ii) : $G$ 0-dimensional

9.1 In this case there is ([7], (7.7)) a base of neighbourhoods of zero in  $G$  formed of compact open subgroups  $W$ . For each such  $W$  the annihilator  $\Delta = W^\circ$  in  $\Gamma$  of  $W$  is a finite subgroup of  $\Gamma$ . Define

$$k_W = \lambda_G(W)^{-1} \times \text{characteristic function of } W. \quad (9.1)$$

Then  $k_W$  is continuous,  $k_W \geq 0$ ,  $\int_G k_W d\lambda_G = 1$ . The transform  $\hat{k}_W$  of  $k_W$  is plainly equal to unity on  $\Delta$ . On the other hand, since  $W$  is a subgroup, we have for  $a \in W$  and  $\gamma \in \Gamma$

$$\begin{aligned} \hat{k}_W(\gamma) &= \int_G k_W(x) \overline{\gamma(x)} d\lambda_G(x) = \int_G k_W(x+a) \overline{\gamma(x)} d\lambda_G(x) \\ &= \int_G k_W(y) \overline{\gamma(y-a)} d\lambda_G(y) \\ &= \gamma(a) \hat{k}_W(\gamma), \end{aligned}$$

which shows that  $\hat{k}_W(\gamma) = 0$  if  $\gamma \in \Gamma \setminus \Delta$ . Thus  $\hat{k}_W$  is the characteristic function of  $\Delta$ , and so

$$k_W = D_{W^\circ}. \quad (9.2)$$

By (9.1) and (9.2), a routine argument shows that, if  $1 \leq p < \infty$  and  $f \in L^p(G)$ , then

$$f = \lim_W S_W \circ f \quad (9.3)$$

in  $L^p(G)$ ; and that (9.3) holds uniformly for any continuous  $f$ .

9.2 PROOF OF 7.4 (ii). If  $\Gamma_0$  is any countably infinite subgroup of  $\Gamma$  we can choose a sequence  $W_j$  of compact open subgroups of  $G$  such that

$W_{j+1} \subseteq W_j$  and  $\Gamma_0 \subseteq \bigcup_{j=1}^{\infty} W_j^\circ$ , where  $W_j^\circ$  is a finite subgroup of  $\Gamma$  and  $W_j^\circ \subseteq W_{j+1}^\circ$ . The  $\Delta_j = W_j^\circ \cap \Gamma_0$  satisfy (7.2) and, from (9.3),

$$f = \lim_j S_{\Delta_j} f \quad (9.4)$$

uniformly for any continuous  $f$  with  $\text{sp}(f) \subseteq \Gamma_0$ . This verifies the statements made in 7.4 (ii).

9.3 By using the results in [3], more can be said in case (ii) of 7.4; cf. [3], Theorem (2.9) and Example (4.8).

Let  $f \in L^1(G)$  and let  $\Gamma_0$  be any countable subgroup of  $\Gamma$  containing  $\text{sp}(f)$ . Choose the  $W_j$  as in 9.2. Then, apart from the fact that  $(W_j)$  is not in general a base at 0 in  $G$  (they can be chosen to be so if and only if  $G$  is first countable),  $(W_j)$  is an open-compact  $D''$ -sequence ([3], p. 188). The proof of Theorem (2.5) of [3] is easily modified to show that

$$f(x) = \lim_{j \rightarrow \infty} S_{W_j^\circ} f(x) \quad (9.5)$$

holds for almost all  $x \in G$ . Moreover, Theorem (2.7) of [3] applies to show that the majorant function

$$S^* f(x) = \sup_{j \in \mathbb{N}} |S_{W_j^\circ} f(x)| \quad (9.6)$$

satisfies the estimates

$$\|S^* f\|_p \leq 2(p(p-1)^{-1})^{\frac{1}{p}} \|f\|_p \quad (1 < p < \infty) \quad (9.7)$$

$$\|S^* f\|_1 \leq 2 + 2 \int_G |f| \log^+ |f| d\lambda_G, \quad (9.8)$$

$$\|S^* f\|_p \leq 2(1-p)^{\frac{1}{p}} \|f\|_1 \quad (0 < p < 1). \quad (9.9)$$

In particular, the convergence in (9.5) is dominated whenever

$$|f| \log^+ |f| \in L^1(G).$$

A more immediate consequence of (9.1) and (9.2) is a strong version of localisability of the convergence of Fourier series: if  $f \in L^1(G)$  vanishes a.e. on some neighbourhood of  $x_0 \in G$ , we can choose the  $W_j$  so that  $S_{\Delta_j} f(x_0) = 0$  for every sufficiently large  $j$ . [A suitable choice of  $W_j$  may be made once for all, independent of  $f$ , if  $G$  is first countable.] Nothing similar is true for general  $G$ ; see, for example, [11], Vol. II, pp. 304-305.

§ 10. Concerning the polynomials  $Q_j$ .

There is no difficulty in making fairly explicit the construction of t.p.s  $Q_j$  of the type employed in 7.6.

For  $p > 0$ ,  $t \geq 0$  define

$$h_p(t) = \begin{cases} 1 & \text{if } t \leq p, \\ 2 \left(1 - \frac{t}{2p}\right) & \text{if } p \leq t \leq 2p, \\ 0 & \text{if } t \geq 2p. \end{cases} \quad (10.1)$$

For all complex  $z$  define

$$f_p(z) = \begin{cases} 0 & \text{if } z = 0, \\ |z|^{-1} \bar{z} h_p(|z|) & \text{if } z \neq 0. \end{cases} \quad (10.2)$$

Write

$$\left. \begin{aligned} E_n(z) &= \pi^{-1} n \exp(-n|z|^2), \\ P_{n,k}(z) &= \pi^{-1} n \sum_{j=0}^k \frac{(-1)^j}{j!} (n|z|^2)^j \end{aligned} \right\} \quad (10.3)$$

Let  $\mu$  denote Lebesgue measure on  $C$  (identified with  $R^2$  in the canonical fashion).

It is then routine to verify that

$$\left. \begin{aligned} \|E_n * f_p\|_{\infty} &\leq \|f_p\|_{\infty} = 1, \\ \lim_{n \rightarrow \infty} E_n * f_p &= f_p \end{aligned} \right\} \quad (10.4)$$

uniformly on any compact set omitting 0. From this it follows that to every  $p > 0$  and every positive integer  $v$  correspond positive integers  $\bar{n}(p, v)$ ,  $\bar{k}(p, v)$  such that

$$\left. \begin{aligned} \left| |z|^{-1} \bar{z} - f_p * P_{\bar{n}, \bar{k}}(z) \right| &\leq \frac{1}{v} \text{ for } \frac{1}{v} \leq |z| \leq p, \\ \left| f_p * P_{\bar{n}, \bar{k}}(z) \right| &\leq 1 + \frac{1}{v} \text{ for } |z| \leq p. \end{aligned} \right\} \quad (10.5)$$

Now

$$f_p * P_{\bar{n}, \bar{k}}(z) = q_{p,v}(z, \bar{z}), \quad (10.6)$$



where

$$q_{p,v}(X, Y) = \pi^{-1} \bar{n}(p, v) \sum_{j=0}^{\bar{k}(p,v)} \frac{(-\bar{n}(p, v))^j}{j!} \sum_{l=0}^j \sum_{m=0}^j \binom{j}{l} \binom{j}{m} X^l Y^m \\ (-1)^{l+m} \int \zeta^{j-l} \bar{\zeta}^{j-m} f_p(\zeta) d\mu(\zeta) \\ = \sum_{l,m=0}^{\bar{k}(p,v)} C_{p,v}(l, m) X^l Y^m. \quad (10.7)$$

It is easily verifiable that the  $C_{p,v}(l, m)$  are real-valued.

If  $\theta$  is a bounded measurable function on  $G$  and

$$Q_{p,v}^\circ = q_{p,v}(\theta, \bar{\theta}), p \geq \|\theta\|_\infty, \quad (10.8)$$

we have from (10.5)

$$\left. \begin{aligned} \left| |\theta|^{-1} \bar{\theta} - Q_{p,v}^\circ \right| &\leq \frac{1}{v} \text{ whenever } |\theta| \geq \frac{1}{v}, \\ \left| Q_{p,v}^\circ \right| &\leq 1 + \frac{1}{v} \text{ everywhere on } G. \end{aligned} \right\} \quad (10.9)$$

If  $\theta$  is a t.p., then  $Q_{p,v}^\circ$  is a t.p. and

$$\text{sp}(Q_{p,v}^\circ) \subseteq [\text{sp}(\theta)]. \quad (10.10)$$

From (10.9) we obtain

$$\left| |\theta| - \theta Q_{p,v}^\circ \right| \leq \begin{cases} v^{-1} |\theta| & \text{whenever } |\theta| \geq \frac{1}{v}, \\ \left(2 + \frac{1}{v}\right) |\theta| & \text{everywhere,} \end{cases}$$

whence it follows that, if  $\theta \neq 0$ ,

$$\left| \int_G \theta Q_{p,v}^\circ d\lambda_G \right| \geq (1 - v^{-1}) \|\theta\|_1 - v^{-1} (2 + v^{-1}) \\ \geq (1 - 2v^{-\frac{1}{2}}) \|\theta\|_1 \quad (10.11)$$

provided  $v \geq 9 \|\theta\|_1^{-2}$ .

Taking  $\theta = D_{\Delta_j}$  and  $p_j \geq \|D_{\Delta_j}\|$ , the trigonometric polynomials

$$Q_j' = \left(1 + \frac{1}{v}\right)^{-1} Q_{p_j,v}^\circ = \left(1 + \frac{1}{v}\right)^{-1} q_{p_j,v}(D_{\Delta_j}, \bar{D}_{\Delta_j}) \quad (10.12)$$

are then seen from (10.9), (10.10) and (10.11) to satisfy

$$\left. \begin{aligned} \| Q_j' \| &\leq 1, \\ \text{sp } (Q_j') &\subseteq [A_j], \\ \left| \int v D_{A_j} Q_j' d\lambda_G \right| &\geq (1 - 3v^{-\frac{1}{2}}) \| D_{A_j} \|_1 \end{aligned} \right\} \quad (10.13)$$

provided  $v$  is chosen  $\geq 9 \| D_{A_j} \|_1^{-1}$ . In view of (7.6), we may choose the integer  $v \geq \max_j (36, 9 \| D_{A_j} \|_1^{-1})$ . Then (10.13) shows that there are unimodular complex numbers  $\xi_j$  such that the  $Q_j = \xi_j Q_j'$  satisfy (7.7).

## APPENDIX

### *Rudin-Shapiro sequences*

**A.1 NOTATIONS AND DEFINITIONS.** As hitherto, all topological groups  $G$  are assumed to be Hausdorff; and, for any locally compact group  $G$ ,  $\lambda_G$  will denote a selected left Haar measure, with respect to which the Lebesgue spaces  $L^p(G)$  are to be formed.  $C_c(G)$  denotes the set of complex-valued continuous functions on  $G$  having compact supports.

If  $X$  and  $Y$  are topological groups,  $\text{Hom } (X, Y)$  denotes the set of continuous homomorphisms of  $X$  into  $Y$ .

Suppose henceforth  $G$  to be locally compact. As in 5.1, if  $k \in C_c(G)$ ,  $T_k$  will denote the convolution operator

$$f \mapsto f * k$$

with domain  $C_c(G)$  and range in  $C_c(G)$ ; and  $\| k \|_{p,q}$  will denote the  $(p, q)$ -norm of this operator, i.e., the smallest real number  $m \geq 0$  such that

$$\| f * k \|_q \leq m \| f \|_p \quad (f \in C_c(G)).$$

It is well-known that, if  $G$  is Abelian,  $\| k \|_{2,2}$  is equal to

$$\| \hat{k} \|_\infty = \sup_{\gamma \in \Gamma} | \hat{k}(\gamma) |,$$

where  $\Gamma$  is the character group of  $G$  and  $\hat{k}$  is the Fourier transform of  $k$ . (Something similar is true whenever  $G$  is compact, but we shall not use this.)

$U$ -RS-sequences on  $G$  are as defined in 5.4.