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(2) Local convexity is needed in the proof of 3.1 since otherwise (2.2''), i.e., the boundedness of  $S = \{x_n : n \in N\}$  in  $E$ , does not guarantee the existence of any continuous or bounded linear map  $T$  from  $l^1(N)$  into  $E$  such that  $S$  is contained in the  $T$ -image of a bounded subset of  $l^1(N)$ . For it is plain that such a  $T$  can exist, only if the convex envelope  $S'$  of  $S$  is bounded in  $E$ . On the other hand, it is not difficult to verify that any first countable linear topological space  $E$ , in which the convex envelope of every bounded set (or of the range of every sequence converging to zero in  $E$ ) is bounded, is necessarily locally convex.

(3) Naturally, local convexity of  $E$  may be dropped from the hypotheses of 3.1, if one assumes in place of (2.2'') that the convex envelope of  $\{x_n : n \in N\}$  is a bounded subset of  $E$ .

#### § 4. *Deduction of boundedness principles*

4.1 THEOREM. Suppose that  $E$  is a sequentially complete locally convex space and that  $P$  is a set of bounded gauges on  $E$ . If  $f^*(x) = \sup \{f(x) : f \in P\} < \infty$  for every  $x \in E$ , then  $f^*$  is bounded.

PROOF. Suppose the contrary, that is, that  $f^*(x) < \infty$  for every  $x \in E$  and yet there exists a bounded subset  $B$  of  $E$  on which  $f^*$  is unbounded. Then we can choose  $x_n \in B$ ,  $f_n \in P$  such that  $f_n(x_n) > n$  for every  $n \in N$ . Then (2.1), (2.2'') and (2.3) are satisfied; hence, by 3.1, there exists  $x \in E$  such that  $f^*(x) = \infty$ , which is the required contradiction.

4.2 REMARKS. (1) If we assume also that  $E$  is infrabarrelled and that each  $f \in P$  is continuous, it follows that  $f^*$  is continuous, that is, that  $P$  is equicontinuous if it is pointwise bounded; cf. [2], pp. 47, 480-81. For, if  $V$  denotes the interval  $[-\varepsilon, \varepsilon]$ , where  $\varepsilon > 0$ , then

$$f^{*-1}(V) = \bigcap \{f^{-1}(V) : f \in P\}$$

is closed, convex and balanced and absorbs bounded sets in  $E$ . Since  $E$  is infrabarrelled,  $f^{*-1}(V)$  is therefore a neighbourhood of the origin in  $E$  and thus  $f^*$  is continuous, as asserted.

(2) If one drops the hypothesis that  $E$  be locally convex (the remaining assumptions of Theorem 4.1 remaining intact), the substance of Remark 3.3 (3) shows that one may still conclude that  $f^*(B)$  is bounded whenever  $B$  is a subset of  $E$  whose convex envelope in  $E$  is bounded.

However, even assuming that  $E$  is first countable and complete, one can in general no longer conclude that  $f^*$  is bounded (i.e., that  $f^*(A)$  is bounded for every bounded subset  $A$  of  $E$ ) whenever it is finite-valued. Counter-examples are easily given in the case of the familiar spaces  $E = l^p(N)$  with  $p \in (0, 1)$ .

## PART 2: APPLICATIONS TO MULTIPLIERS

### § 5. $(p, q)$ -multipliers which are not measures

5.1 INTRODUCTION. In this section and the following one we will use the substance of § 3 to prove several apparently new properties of  $(p, q)$ -multipliers. Let  $G$  be a locally compact group [all topological groups will be assumed to be Hausdorff and, in this section, will be multiplicatively written with identity  $e$ ]. Denote by  $L^p(G)$ , where  $1 \leq p \leq \infty$ , the usual Lebesgue space formed with a fixed left Haar measure  $\lambda_G$  on  $G$ ; and by  $C_c(G)$  the space of continuous complex-valued functions on  $G$  with compact supports.

For  $a \in G$ , define the left translation operator  $\tau_a$  and the right translation operator  $\rho_a$  by

$$\tau_a g(x) = g(a^{-1}x) \quad \text{and} \quad \rho_a g(x) = g(xa^{-1});$$

respectively. A linear operator  $T$  from  $C_c(G)$  into  $L^q(G)$  is said to be a (left)  $(p, q)$ -multiplier if and only if

- (i)  $T$  is continuous from  $C_c(G)$ , equipped with the norm induced by  $L^p(G)$ , into  $L^q(G)$ ; and
- (ii)  $T$  commutes with left translations, that is  $T\tau_a = \tau_a T$  for all  $a \in G$ .

A right  $(p, q)$ -multiplier is defined in a similar manner with (ii) replaced by

$$(ii') \quad T\rho_a = \rho_a T \text{ for all } a \in G.$$

Let  $L_p^q(G)$  denote the Banach space of  $(p, q)$ -multipliers equipped with the customary norm, denoted by  $\|\cdot\|_{p,q}$ , of continuous linear operators from a subspace of  $L^p(G)$  into  $L^q(G)$ . That is, for each  $T \in L_p^q(G)$ ,  $\|T\|_{p,q}$  is the smallest real number  $K$  satisfying

$$\|Tg\|_q \leq K \|g\|_p$$