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for every sequence  $(\gamma_n)$  satisfying (2.4) the series

$$\sum_{n \in N} \gamma_n \lambda_n x_n \tag{2.14}$$

is normally convergent in  $E$ ; and

$$f^*(x) = \infty \tag{2.15}$$

for every sum  $x$  of the series (2.14).

In the sequel we shall denote by  $l_+^1(N)$  the set of sequences  $(\lambda_n)$  satisfying (2.13).

PROOF. Define by recurrence a strictly increasing sequence  $(k_n)$  of positive integers, taking  $k_1$  to be the first  $k \in N$  such that  $f_k(x_k) > 1^3$  and  $k_{n+1}$  to be the first  $k \in N$  such that  $k > k_n$  and  $f_k(x_k) > (n+1)^3$ . Then apply 2.1 and 2.2 with  $x_n$  and  $f_n$  replaced by  $n^{-2} x_{k_n}$  and  $f_{k_n}$  respectively. This furnishes at least one strictly increasing sequence  $(n_v)$  of positive integers such that (2.4) entails that the series

$$\sum_{v \in N} \gamma_v n_v^{-2} x_{k_{n_v}} \tag{2.16}$$

is normally convergent in  $E$  and that (2.15) holds for every sum  $x$  of (2.16). It thus suffices to define  $\lambda_n$  to be  $n_v^{-2}$  when  $n = k_{n_v}$  for some  $v \in N$  and to be zero for all other  $n \in N$ ; it is obvious that (2.13) is then satisfied.

### § 3. The construction when $E$ is sequentially complete

3.1 In this section we assume merely that  $E$  is a locally convex space which is sequentially complete. Again  $P$  will denote a set of bounded gauges on  $E$ , and  $f^*$  will denote its upper envelope. Suppose given sequences  $(x_n)$  in  $E$  and  $(f_n)$  in  $P$  such that (2.1), (2.2'') and (2.3) are satisfied. Then the conclusion of 2.4 remains valid.

PROOF. Consider the continuous linear map  $T$  of  $l^1(N)$  into  $E$  defined by

$$T\xi = \sum_{n \in N} \xi_n x_n.$$

Evidently,  $x_n = T\alpha_n$  for suitably chosen  $\alpha_n$  such that  $\{\alpha_n : n \in N\}$  is a bounded subset of  $l^1(N)$ . It therefore suffices to apply 2.4 with  $E$  replaced by  $l^1(N)$ ,  $x_n$  by  $\alpha_n$ , and  $f_n$  by  $f_n \circ T$ .

The following corollary will find application in §§ 5 and 6 below.

3.2 COROLLARY. Suppose that  $H$  is a Hausdorff topological linear space and that  $(E_i)_{i \in I}$  is a family of linear subspaces of  $H$  such that

- (i)  $E_i$  is a Banach space relative to a norm  $\| \cdot \|_i$  and the injection  $E_i \rightarrow H$  is continuous.

Let  $\mathcal{E} = \bigcap \{E_i : i \in I\}$  be topologised as a topological linear space by taking a base at 0 in  $\mathcal{E}$  formed of the sets  $\{x \in \mathcal{E} : \sup_{i \in J} \|x\|_i < \varepsilon\}$ , where  $\varepsilon$  ranges over positive numbers and  $J$  over finite subsets of  $I$ . Let  $E$  be a sequentially closed linear subspace of  $\mathcal{E}$  and  $(f_n)_{n \in N}$  a sequence of bounded gauges on  $E$ , and write  $f^*$  for the upper envelope of  $(f_n)_{n \in N}$ . Suppose finally that  $(x_n)_{n \in N}$  is a sequence of elements of  $E$  such that

- (ii)  $f^*(x_n) < \infty$  for every  $n \in N$ ;  
 (iii)  $\sup_{n \in N} \|x_n\|_i < \infty$  for every  $i \in I$ ;  
 (iv)  $\sup_{n \in N} f_n(x_n) = \infty$ .

The conclusion is that, given real numbers  $\beta > \alpha > 0$ , a sequence  $(\lambda_n)_{n \in N} \in l_+^1(N)$  may be constructed such that, for every sequence  $(\gamma_n)_{n \in N}$  satisfying (2.4), the series (2.14) is normally convergent in  $E$  to a (unique) sum  $x$  satisfying (2.15).

PROOF. In view of 3.1, it will suffice to verify that  $\mathcal{E}$  (which is obviously locally convex) is sequentially complete and Hausdorff. The latter property is evidently present. As to the former, suppose that  $(y_n)_{n \in N}$  is a Cauchy sequence in  $\mathcal{E}$ . Then, by definition of the topology on  $\mathcal{E}$ ,  $(y_n)$  is Cauchy in  $E_i$  for every  $i \in I$ . Hence, by the first clause of (i),  $(y_n)$  is convergent in  $E_i$  to a limit  $y_{(i)} \in E_i$ . The second clause of (i), plus the fact that  $H$  is Hausdorff, entails that there exists  $y \in H$  such that  $y_{(i)} = y$  for every  $i \in I$ . Accordingly,  $y \in \mathcal{E}$ ; and, since  $\lim_{n \rightarrow \infty} y_n = y_{(i)} = y$  in  $E_i$  for every  $i \in I$ ,  $\lim_{n \rightarrow \infty} y_n = y$  in  $\mathcal{E}$ . This shows that  $\mathcal{E}$  is sequentially complete.

3.3 REMARKS. (1) If the elements of  $P$  are seminorms (rather than merely gauges), we may everywhere permit  $(\gamma_n)$  to be a sequence taking values in the (real or complex) scalar field of  $E$ , replacing (2.4) by the condition

$$\alpha \leq |\gamma_n| \leq \beta \quad \text{for every } n \in N. \quad (2.4')$$

This is easily seen by reverting to 2.2 and using the fact that now  $f_n(\gamma x) = |\gamma| f_n(x)$  for every  $x \in E$ , every  $n \in N$  and every scalar  $\gamma$ . No changes are needed in the choice of the  $n_\nu$ .

(2) Local convexity is needed in the proof of 3.1 since otherwise (2.2''), i.e., the boundedness of  $S = \{x_n : n \in N\}$  in  $E$ , does not guarantee the existence of any continuous or bounded linear map  $T$  from  $l^1(N)$  into  $E$  such that  $S$  is contained in the  $T$ -image of a bounded subset of  $l^1(N)$ . For it is plain that such a  $T$  can exist, only if the convex envelope  $S'$  of  $S$  is bounded in  $E$ . On the other hand, it is not difficult to verify that any first countable linear topological space  $E$ , in which the convex envelope of every bounded set (or of the range of every sequence converging to zero in  $E$ ) is bounded, is necessarily locally convex.

(3) Naturally, local convexity of  $E$  may be dropped from the hypotheses of 3.1, if one assumes in place of (2.2'') that the convex envelope of  $\{x_n : n \in N\}$  is a bounded subset of  $E$ .

#### § 4. *Deduction of boundedness principles*

4.1 THEOREM. Suppose that  $E$  is a sequentially complete locally convex space and that  $P$  is a set of bounded gauges on  $E$ . If  $f^*(x) = \sup \{f(x) : f \in P\} < \infty$  for every  $x \in E$ , then  $f^*$  is bounded.

PROOF. Suppose the contrary, that is, that  $f^*(x) < \infty$  for every  $x \in E$  and yet there exists a bounded subset  $B$  of  $E$  on which  $f^*$  is unbounded. Then we can choose  $x_n \in B$ ,  $f_n \in P$  such that  $f_n(x_n) > n$  for every  $n \in N$ . Then (2.1), (2.2'') and (2.3) are satisfied; hence, by 3.1, there exists  $x \in E$  such that  $f^*(x) = \infty$ , which is the required contradiction.

4.2 REMARKS. (1) If we assume also that  $E$  is infrabarrelled and that each  $f \in P$  is continuous, it follows that  $f^*$  is continuous, that is, that  $P$  is equicontinuous if it is pointwise bounded; cf. [2], pp. 47, 480-81. For, if  $V$  denotes the interval  $[-\varepsilon, \varepsilon]$ , where  $\varepsilon > 0$ , then

$$f^{*-1}(V) = \bigcap \{f^{-1}(V) : f \in P\}$$

is closed, convex and balanced and absorbs bounded sets in  $E$ . Since  $E$  is infrabarrelled,  $f^{*-1}(V)$  is therefore a neighbourhood of the origin in  $E$  and thus  $f^*$  is continuous, as asserted.

(2) If one drops the hypothesis that  $E$  be locally convex (the remaining assumptions of Theorem 4.1 remaining intact), the substance of Remark 3.3 (3) shows that one may still conclude that  $f^*(B)$  is bounded whenever  $B$  is a subset of  $E$  whose convex envelope in  $E$  is bounded.