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Our final preliminary comment refers to boundedness of sets. If  $E$  is any topological linear space, a subset  $A$  of  $E$  will be said to be bounded in  $E$  if and only if to every neighbourhood  $U$  of 0 in  $E$  corresponds a number  $r = r(A, U) > 0$  such that  $rA = \{rx : x \in A\}$  is contained in  $U$ . If  $E$  is first countable and  $d$  is a semimetric on  $E$  defining its topology, boundedness in the above sense of a set  $A \subseteq E$  must not be confused with metric boundedness [i.e., with the condition  $\sup \{d(x, y) : x \in A, y \in A\} < \infty$ ]. It is in order to minimise the possibility of this confusion that we use the term “first countable” (an abbreviation for “satisfying the first axiom of countability”) rather than “semimetrizable”.

§ 2. *The construction when  $E$  is complete and first countable.*

In this section, where  $E$  will always denote a complete first countable (locally convex) space and  $P$  a set of bounded gauges on  $E$ , we will describe the basic construction. Let  $f^*$  denote the upper envelope of  $P$ .

If the sequence  $(x_n)$  figuring in (1.1) and (1.2) is such that  $f^*(x_n) = \infty$  for some  $n \in N$ , no constructional problem remains. So we shall henceforth assume the contrary.

2.1 THEOREM. Suppose that  $\beta$  and  $\alpha$  are real numbers satisfying  $\beta > \alpha > 0$  and that sequences  $(x_n)$  in  $E$ ,  $(f_n)$  in  $P$  are such that:

$$f^*(x_n) < \infty \quad \text{for every } n \in N, \quad (2.1)$$

$$\lim_{n \rightarrow \infty} x_n = 0, \quad (2.2)$$

$$\sup_{n \in N} f_n(x_n) = \infty. \quad (2.3)$$

Then infinite sequences  $n_1 < n_2 < \dots$  of positive integers may be constructed such that, for every sequence  $(\gamma_n)$  of real numbers satisfying

$$\alpha \leq \gamma_n \leq \beta \quad \text{for every } n \in N, \quad (2.4)$$

the series

$$\sum_{v \in N} \gamma_v x_{n_v} \quad (2.5)$$

is normally convergent in  $E$ , and

$$f^*(x) \geq \lim_{v \rightarrow \infty} f_{n_v}(x) = \infty \quad (2.6)$$

for each sum  $x$  of (2.5).

2.2 CONSTRUCTION AND PROOF. Let  $(\sigma_v)$  be an increasing sequence of continuous seminorms on  $E$  which define its topology. By initial passage to suitable subsequences, we may and will assume that (2.2) and (2.3) hold in the stronger form:

$$\sum_{n \in N} \sigma_n(x_n) < \infty, \quad (2.2')$$

$$\lim_{n \rightarrow \infty} f_n(x_n) = \infty. \quad (2.3')$$

[To do this, define  $n_v \in N$  for  $v \in N$  by induction in such a way that  $n_1 < n_2 < \dots$ ,

$$\sigma_v(x_{n_v}) \leq 2^{-v} \quad \text{and} \quad f_{n_v}(x_{n_v}) > v \quad (2.7)$$

for all  $v \in N$ . This is possible since by (2.2) we can determine  $n_1^\circ \in N$  such that  $\sigma_1(x_n) \leq 2^{-1}$  if  $n \geq n_1^\circ$ , and then, by (2.3) and the fact that each  $f \in P$  is finite valued, there exists  $n \geq n_1^\circ$  such that  $f_n(x_n) > 1$ ; denote the smallest such  $n \geq n_1^\circ$  by  $n_1$ . When  $n_1 < n_2 < \dots < n_j$  have been determined so that (2.7) holds for  $1 \leq v \leq j$ , find (see (2.2)) an integer  $n_{j+1}^\circ > n_j$  such that  $\sigma_{j+1}(x_n) \leq 2^{-j-1}$  if  $n \geq n_{j+1}^\circ$ . Then (2.3) shows that there exists an integer  $n \geq n_{j+1}^\circ$  such that  $f_n(x_n) > j+1$ ; put  $n_{j+1}$  for the smallest such integer  $n \geq n_{j+1}^\circ$ .]

So now we assume (2.1), (2.2') and (2.3') and define one sequence  $n_1 < n_2 < \dots$  of the required type in the following manner. (Other possibilities are discussed in Remark 2.3 (2) below.) Let  $n_1$  be the smallest  $n \in N$  such that

$$f_n(x_n) \geq \beta\alpha^{-1};$$

$n_1$  may be determined by (2.3'). Suppose that  $v$  is a positive integer and that positive integers  $n_1 < n_2 < \dots < n_v$  have been defined so that

$$f_{n_j}(x_{n_v}) \leq 2^{-v} \quad \text{whenever} \quad 1 \leq j < v,$$

$$f_{n_v}(x_{n_v}) \geq \beta\alpha^{-1} \sum_{1 \leq j < v} f_{n_j}(x_{n_j}) + \beta\alpha^{-1} v.$$

[An empty sum is defined to be 0; then the conditions are all satisfied when  $v = 1$ .] Then (2.2'), (2.3') and the fact that each  $f \in P$  is finite-valued imply that there exists an integer  $n > n_v$  which satisfies

$$f_{n_j}(x_n) \leq 2^{-v-1} \quad \text{whenever} \quad 1 \leq j < v+1,$$

$$f_n(x_n) \geq \beta\alpha^{-1} \sum_{1 \leq j < v+1} f_{n_j}(x_{n_j}) + \beta\alpha^{-1} (v+1);$$

let  $n_{v+1}$  be the smallest such  $n$ . We then have for each  $v \in N$ :

$$n_v < n_{v+1},$$

$$f_{n_j}(x_{n_v}) \leq 2^{-v} \quad \text{whenever } 1 \leq j < v, \quad (2.8)$$

$$f_{n_v}(x_{n_v}) \geq \beta \alpha^{-1} \sum_{1 \leq j < v} f_{n_j}(x_{n_j}) + \beta \alpha^{-1} v. \quad (2.9)$$

By (2.2') and (2.4), the sum (2.5) is normally convergent in  $E$ . Let  $x$  be any sum of this series. To establish (2.6), write

$$x = u_v + \gamma_v x_{n_v} + v_v,$$

where  $u_v = \sum_{1 \leq j < v} \gamma_j x_{n_j}$  and  $v_v$  is a sum of the series  $\sum_{j > v} \gamma_j x_{n_j}$ . Thus  $\gamma_v x_{n_v} = x - u_v - v_v$ , and so

$$\alpha f_{n_v}(x_{n_v}) \leq f_{n_v}(\gamma_v x_{n_v}) \leq f_{n_v}(x) + f_{n_v}(u_v) + f_{n_v}(v_v). \quad (2.10)$$

Now, by (2.4),

$$f_{n_v}(u_v) \leq \beta \sum_{1 \leq j < v} f_{n_j}(x_{n_j}); \quad (2.11)$$

and, by (2.4), (2.8) and the fact that each  $f_n$  is bounded, hence continuous,

$$f_{n_v}(v_v) \leq \beta \sum_{j > v} f_{n_j}(x_{n_j}) \leq \beta \sum_{j > v} 2^{-j} = \beta 2^{-v}. \quad (2.12)$$

By (2.10), (2.11) and (2.12)

$$\alpha f_{n_v}(x_{n_v}) \leq f_{n_v}(x) + \beta \sum_{1 \leq j < v} f_{n_j}(x_{n_j}) + \beta 2^{-v},$$

and so, by (2.9),

$$\beta \sum_{1 \leq j < v} f_{n_j}(x_{n_j}) + \beta v \leq f_{n_v}(x) + \beta \sum_{1 \leq j < v} f_{n_j}(x_{n_j}) + \beta 2^{-v}.$$

Hence

$$f_{n_v}(x) \geq \beta(v - 2^{-v}),$$

which proves (2.6) and the construction is complete.

### 2.3 REMARKS. (1) If it is known that

$$D = \{x \in E : f^*(x) < \infty\}$$

is dense in  $E$ , and if  $(x_n)$  and  $(f_n)$  satisfy (2.2) and (2.3), we can approximate each  $x_n$  so closely by an element  $y_n$  of  $D$  that (2.2) and (2.3) are left intact on replacing  $x_n$  by  $y_n$ . The hypotheses (2.1)–(2.3) are satisfied when  $x_n$  is everywhere replaced by  $y_n$ .

(2) If it be supposed that (2.2') holds and that sequences  $(A_n)$ ,  $(B_{n,r})$  and  $(C_n)$  are known such that  $\lim_{n \rightarrow \infty} B_{n,r} = 0$  for every  $r \in N$ ,  $\lim_{n \rightarrow \infty} C_n = \infty$ ,

$$f^*(x_1) + \dots + f^*(x_n) \leq A_n,$$

$$\max_{1 \leq j \leq r} f_j(x_n) \leq B_{n,r},$$

$$f_n(x_n) \geq C_n,$$

then it is easy to specify a function  $\phi_{\alpha,\beta} : N \times N \rightarrow N$  in terms of  $(A_n)$ ,  $(B_{n,r})$  and  $(C_n)$  such that (2.4) and (2.5) yield (2.6) for every sequence  $(n_v)$  such that  $C_{n_1} \geq \beta\alpha^{-1}$  and  $n_{v+1} \geq \phi_{\alpha,\beta}(n_v, v)$  for every  $v \in N$ .

(3) Local convexity of  $E$  is not essential in 2.1 and 2.2. In the contrary case one may proceed by introducing an invariant semimetric  $(x, y) \mapsto |x - y|$  defining the topology of  $E$ , much as in [2], proof of Theorem 6.1.1, or [15], Chapitre I, § 3, No. 1. Normal summability in  $E$  of a series  $\sum_{n \in N} z_n$  of elements of  $E$  may then be taken to mean the convergence of  $\sum_{n \in N} |z_n|$ . In place of (2.2') arrange that

$$\sum_{n \in N} |\beta x_n| < \infty,$$

which will ensure the normal convergence in  $E$  of (2.5) whenever (2.4) holds ( $E$  being assumed to be complete). The rest of the proof and construction proceeds as before.

This method could, of course, be used when  $E$  is locally convex (and first countable and complete); we have not done so because the seminorms  $\sigma_n$  are usually more manageable in practice.

(4) A useful variant of 2.1 may be stated in the following terms.

2.4 Suppose given real numbers  $\beta > \alpha > 0$  and sequences  $(x_n)$  in  $E$  and  $(f_n)$  in  $P$  such that

$$f^*(x_n) < \infty \quad \text{for every } n \in N, \quad (2.1)$$

$$\{x_n : n \in N\} \quad \text{is bounded in } E, \quad (2.2'')$$

$$\sup_{n \in N} f_n(x_n) = \infty. \quad (2.3)$$

Then one can construct a sequence  $(\lambda_n)$  of real numbers with the following properties:

$$\lambda_n \geq 0, \quad \sum_{n \in N} \lambda_n < \infty; \quad (2.13)$$

for every sequence  $(\gamma_n)$  satisfying (2.4) the series

$$\sum_{n \in N} \gamma_n \lambda_n x_n \quad (2.14)$$

is normally convergent in  $E$ ; and

$$f^*(x) = \infty \quad (2.15)$$

for every sum  $x$  of the series (2.14).

In the sequel we shall denote by  $l_+^1(N)$  the set of sequences  $(\lambda_n)$  satisfying (2.13).

PROOF. Define by recurrence a strictly increasing sequence  $(k_n)$  of positive integers, taking  $k_1$  to be the first  $k \in N$  such that  $f_k(x_k) > 1^3$  and  $k_{n+1}$  to be the first  $k \in N$  such that  $k > k_n$  and  $f_k(x_k) > (n+1)^3$ . Then apply 2.1 and 2.2 with  $x_n$  and  $f_n$  replaced by  $n^{-2} x_{k_n}$  and  $f_{k_n}$  respectively. This furnishes at least one strictly increasing sequence  $(n_v)$  of positive integers such that (2.4) entails that the series

$$\sum_{v \in N} \gamma_v n_v^{-2} x_{k_{n_v}} \quad (2.16)$$

is normally convergent in  $E$  and that (2.15) holds for every sum  $x$  of (2.16). It thus suffices to define  $\lambda_n$  to be  $n_v^{-2}$  when  $n = k_{n_v}$  for some  $v \in N$  and to be zero for all other  $n \in N$ ; it is obvious that (2.13) is then satisfied.

### § 3. The construction when $E$ is sequentially complete

3.1 In this section we assume merely that  $E$  is a locally convex space which is sequentially complete. Again  $P$  will denote a set of bounded gauges on  $E$ , and  $f^*$  will denote its upper envelope. Suppose given sequences  $(x_n)$  in  $E$  and  $(f_n)$  in  $P$  such that (2.1), (2.2'') and (2.3) are satisfied. Then the conclusion of 2.4 remains valid.

PROOF. Consider the continuous linear map  $T$  of  $l^1(N)$  into  $E$  defined by

$$T\xi = \sum_{n \in N} \xi_n x_n.$$

Evidently,  $x_n = T\alpha_n$  for suitably chosen  $\alpha_n$  such that  $\{\alpha_n : n \in N\}$  is a bounded subset of  $l^1(N)$ . It therefore suffices to apply 2.4 with  $E$  replaced by  $l^1(N)$ ,  $x_n$  by  $\alpha_n$ , and  $f_n$  by  $f_n \circ T$ .

The following corollary will find application in §§ 5 and 6 below.