

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 16 (1970)
Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: NAIVELY CONSTRUCTIVE APPROACH TO BOUNDEDNESS PRINCIPLES, WITH APPLICATIONS TO HARMONIC ANALYSIS
Autor: Edwards, R. E. / Price, J. F.
Kapitel: § 2. The construction when E is complete and first countable.
DOI: <https://doi.org/10.5169/seals-43866>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 15.01.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Our final preliminary comment refers to boundedness of sets. If E is any topological linear space, a subset A of E will be said to be bounded in E if and only if to every neighbourhood U of 0 in E corresponds a number $r = r(A, U) > 0$ such that $rA = \{rx : x \in A\}$ is contained in U . If E is first countable and d is a semimetric on E defining its topology, boundedness in the above sense of a set $A \subseteq E$ must not be confused with metric boundedness [i.e., with the condition $\sup \{d(x, y) : x \in A, y \in A\} < \infty$]. It is in order to minimise the possibility of this confusion that we use the term “first countable” (an abbreviation for “satisfying the first axiom of countability”) rather than “semimetrizable”.

§ 2. The construction when E is complete and first countable.

In this section, where E will always denote a complete first countable (locally convex) space and P a set of bounded gauges on E , we will describe the basic construction. Let f^* denote the upper envelope of P .

If the sequence (x_n) figuring in (1.1) and (1.2) is such that $f^*(x_n) = \infty$ for some $n \in N$, no constructional problem remains. So we shall henceforth assume the contrary.

2.1 THEOREM. Suppose that β and α are real numbers satisfying $\beta > \alpha > 0$ and that sequences (x_n) in E , (f_n) in P are such that:

$$f^*(x_n) < \infty \quad \text{for every } n \in N, \quad (2.1)$$

$$\lim_{n \rightarrow \infty} x_n = 0, \quad (2.2)$$

$$\sup_{n \in N} f_n(x_n) = \infty. \quad (2.3)$$

Then infinite sequences $n_1 < n_2 < \dots$ of positive integers may be constructed such that, for every sequence (γ_n) of real numbers satisfying

$$\alpha \leq \gamma_n \leq \beta \quad \text{for every } n \in N, \quad (2.4)$$

the series

$$\sum_{v \in N} \gamma_v x_{n_v} \quad (2.5)$$

is normally convergent in E , and

$$f^*(x) \geq \lim_{v \rightarrow \infty} f_{n_v}(x) = \infty \quad (2.6)$$

for each sum x of (2.5).

2.2 CONSTRUCTION AND PROOF. Let (σ_v) be an increasing sequence of continuous seminorms on E which define its topology. By initial passage to suitable subsequences, we may and will assume that (2.2) and (2.3) hold in the stronger form:

$$\sum_{n \in N} \sigma_n(x_n) < \infty, \quad (2.2')$$

$$\lim_{n \rightarrow \infty} f_n(x_n) = \infty. \quad (2.3')$$

[To do this, define $n_v \in N$ for $v \in N$ by induction in such a way that $n_1 < n_2 < \dots$,

$$\sigma_v(x_{n_v}) \leq 2^{-v} \quad \text{and} \quad f_{n_v}(x_{n_v}) > v \quad (2.7)$$

for all $v \in N$. This is possible since by (2.2) we can determine $n_1^\circ \in N$ such that $\sigma_1(x_n) \leq 2^{-1}$ if $n \geq n_1^\circ$, and then, by (2.3) and the fact that each $f \in P$ is finite valued, there exists $n \geq n_1^\circ$ such that $f_n(x_n) > 1$; denote the smallest such $n \geq n_1^\circ$ by n_1 . When $n_1 < n_2 < \dots < n_j$ have been determined so that (2.7) holds for $1 \leq v \leq j$, find (see (2.2)) an integer $n_{j+1}^\circ > n_j$ such that $\sigma_{j+1}(x_n) \leq 2^{-j-1}$ if $n \geq n_{j+1}^\circ$. Then (2.3) shows that there exists an integer $n \geq n_{j+1}^\circ$ such that $f_n(x_n) > j+1$; put n_{j+1} for the smallest such integer $n \geq n_{j+1}^\circ$.]

So now we assume (2.1), (2.2') and (2.3') and define one sequence $n_1 < n_2 < \dots$ of the required type in the following manner. (Other possibilities are discussed in Remark 2.3 (2) below.) Let n_1 be the smallest $n \in N$ such that

$$f_n(x_n) \geq \beta \alpha^{-1};$$

n_1 may be determined by (2.3'). Suppose that v is a positive integer and that positive integers $n_1 < n_2 < \dots < n_v$ have been defined so that

$$f_{n_j}(x_{n_v}) \leq 2^{-v} \quad \text{whenever} \quad 1 \leq j < v,$$

$$f_{n_v}(x_{n_v}) \geq \beta \alpha^{-1} \sum_{1 \leq j < v} f_{n_j}(x_{n_j}) + \beta \alpha^{-1} v.$$

[An empty sum is defined to be 0; then the conditions are all satisfied when $v = 1$.] Then (2.2'), (2.3') and the fact that each $f \in P$ is finite-valued imply that there exists an integer $n > n_v$ which satisfies

$$f_{n_j}(x_n) \leq 2^{-v-1} \quad \text{whenever} \quad 1 \leq j < v+1,$$

$$f_n(x_n) \geq \beta \alpha^{-1} \sum_{1 \leq j < v+1} f_{n_j}(x_{n_j}) + \beta \alpha^{-1} (v+1);$$

let n_{v+1} be the smallest such n . We then have for each $v \in N$:

$$n_v < n_{v+1},$$

$$f_{n_j}(x_{n_v}) \leq 2^{-v} \quad \text{whenever} \quad 1 \leq j < v, \quad (2.8)$$

$$f_{n_v}(x_{n_v}) \geq \beta \alpha^{-1} \sum_{1 \leq j < v} f_{n_j}(x_{n_j}) + \beta \alpha^{-1} v. \quad (2.9)$$

By (2.2') and (2.4), the sum (2.5) is normally convergent in E . Let x be any sum of this series. To establish (2.6), write

$$x = u_v + \gamma_v x_{n_v} + v_v,$$

where $u_v = \sum_{1 \leq j < v} \gamma_j x_{n_j}$ and v_v is a sum of the series $\sum_{j > v} \gamma_j x_{n_j}$. Thus $\gamma_v x_{n_v} = x - u_v - v_v$, and so

$$\alpha f_{n_v}(x_{n_v}) \leq f_{n_v}(\gamma_v x_{n_v}) \leq f_{n_v}(x) + f_{n_v}(u_v) + f_{n_v}(v_v). \quad (2.10)$$

Now, by (2.4),

$$f_{n_v}(u_v) \leq \beta \sum_{1 \leq j < v} f_{n_j}(x_{n_j}); \quad (2.11)$$

and, by (2.4), (2.8) and the fact that each f_n is bounded, hence continuous,

$$f_{n_v}(v_v) \leq \beta \sum_{j > v} f_{n_j}(x_{n_j}) \leq \beta \sum_{j > v} 2^{-j} = \beta 2^{-v}. \quad (2.12)$$

By (2.10), (2.11) and (2.12)

$$\alpha f_{n_v}(x_{n_v}) \leq f_{n_v}(x) + \beta \sum_{1 \leq j < v} f_{n_j}(x_{n_j}) + \beta 2^{-v},$$

and so, by (2.9),

$$\beta \sum_{1 \leq j < v} f_{n_j}(x_{n_j}) + \beta v \leq f_{n_v}(x) + \beta \sum_{1 \leq j < v} f_{n_j}(x_{n_j}) + \beta 2^{-v}.$$

Hence

$$f_{n_v}(x) \geq \beta (v - 2^{-v}),$$

which proves (2.6) and the construction is complete.

2.3 REMARKS. (1) If it is known that

$$D = \{x \in E : f^*(x) < \infty\}$$

is dense in E , and if (x_n) and (f_n) satisfy (2.2) and (2.3), we can approximate each x_n so closely by an element y_n of D that (2.2) and (2.3) are left intact on replacing x_n by y_n . The hypotheses (2.1)—(2.3) are satisfied when x_n is everywhere replaced by y_n .

(2) If it be supposed that (2.2') holds and that sequences (A_n) , $(B_{n,r})$ and (C_n) are known such that $\lim_{n \rightarrow \infty} B_{n,r} = 0$ for every $r \in N$, $\lim_{n \rightarrow \infty} C_n = \infty$,

$$f^*(x_1) + \dots + f^*(x_n) \leq A_n,$$

$$\max_{1 \leq j \leq r} f_j(x_n) \leq B_{n,r},$$

$$f_n(x_n) \geq C_n,$$

then it is easy to specify a function $\phi_{\alpha,\beta} : N \times N \rightarrow N$ in terms of (A_n) , $(B_{n,r})$ and (C_n) such that (2.4) and (2.5) yield (2.6) for every sequence (n_v) such that $C_{n_1} \geq \beta\alpha^{-1}$ and $n_{v+1} \geq \phi_{\alpha,\beta}(n_v, v)$ for every $v \in N$.

(3) Local convexity of E is not essential in 2.1 and 2.2. In the contrary case one may proceed by introducing an invariant semimetric $(x, y) \mapsto |x - y|$ defining the topology of E , much as in [2], proof of Theorem 6.1.1, or [15], Chapitre I, § 3, No. 1. Normal summability in E of a series $\sum_{n \in N} z_n$ of elements of E may then be taken to mean the convergence of $\sum_{n \in N} |z_n|$. In place of (2.2') arrange that

$$\sum_{n \in N} |\beta x_n| < \infty,$$

which will ensure the normal convergence in E of (2.5) whenever (2.4) holds (E being assumed to be complete). The rest of the proof and construction proceeds as before.

This method could, of course, be used when E is locally convex (and first countable and complete); we have not done so because the seminorms σ_n are usually more manageable in practice.

(4) A useful variant of 2.1 may be stated in the following terms.

2.4 Suppose given real numbers $\beta > \alpha > 0$ and sequences (x_n) in E and (f_n) in P such that

$$f^*(x_n) < \infty \quad \text{for every } n \in N, \tag{2.1}$$

$$\{x_n : n \in N\} \text{ is bounded in } E, \tag{2.2''}$$

$$\sup_{n \in N} f_n(x_n) = \infty. \tag{2.3}$$

Then one can construct a sequence (λ_n) of real numbers with the following properties:

$$\lambda_n \geq 0, \sum_{n \in N} \lambda_n < \infty; \tag{2.13}$$

for every sequence (γ_n) satisfying (2.4) the series

$$\sum_{n \in N} \gamma_n \lambda_n x_n \quad (2.14)$$

is normally convergent in E ; and

$$f^*(x) = \infty \quad (2.15)$$

for every sum x of the series (2.14).

In the sequel we shall denote by $l_+^1(N)$ the set of sequences (λ_n) satisfying (2.13).

PROOF. Define by recurrence a strictly increasing sequence (k_n) of positive integers, taking k_1 to be the first $k \in N$ such that $f_k(x_k) > 1^3$ and k_{n+1} to be the first $k \in N$ such that $k > k_n$ and $f_k(x_k) > (n+1)^3$. Then apply 2.1 and 2.2 with x_n and f_n replaced by $n^{-2} x_{k_n}$ and f_{k_n} respectively. This furnishes at least one strictly increasing sequence (n_v) of positive integers such that (2.4) entails that the series

$$\sum_{v \in N} \gamma_v n_v^{-2} x_{k_{n_v}} \quad (2.16)$$

is normally convergent in E and that (2.15) holds for every sum x of (2.16). It thus suffices to define λ_n to be n_v^{-2} when $n = k_{n_v}$ for some $v \in N$ and to be zero for all other $n \in N$; it is obvious that (2.13) is then satisfied.

§ 3. The construction when E is sequentially complete

3.1 In this section we assume merely that E is a locally convex space which is sequentially complete. Again P will denote a set of bounded gauges on E , and f^* will denote its upper envelope. Suppose given sequences (x_n) in E and (f_n) in P such that (2.1), (2.2'') and (2.3) are satisfied. Then the conclusion of 2.4 remains valid.

PROOF. Consider the continuous linear map T of $l^1(N)$ into E defined by

$$T\xi = \sum_{n \in N} \xi_n x_n.$$

Evidently, $x_n = T\alpha_n$ for suitably chosen α_n such that $\{\alpha_n : n \in N\}$ is a bounded subset of $l^1(N)$. It therefore suffices to apply 2.4 with E replaced by $l^1(N)$, x_n by α_n , and f_n by $f_n \circ T$.

The following corollary will find application in §§ 5 and 6 below.