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A NAIVELY CONSTRUCTIVE APPROACH TO BOUNDEDNESS PRINCIPLES, WITH APPLICATIONS TO HARMONIC ANALYSIS

by R. E. Edwards and J. F. Price

General Introduction

This paper is partly pedagogical and expository. Thus Part 1 (§§ 1-4) presents a naively constructive approach to boundedness principles. Although this construction leads to results differing but slightly from the standard versions, we feel that this approach (which can be followed with no overt reference to category, barrelled spaces, and so on) offers some pedagogical and expository advantages. We emphasise that the level of constructivity is naive and not fundamental.

The remainder of the paper consists of applications of the constructive procedure. In Part 2 (§§ 5, 6) the applications yield improvements of recent results due to Price and to Gaudry concerning multipliers. In Part 3 (§§ 7-10) the applications are to convergence and divergence of Fourier series of continuous functions on compact Abelian groups. These results (which may be known to the afficionados but which, as far as we know, have not been published hitherto) characterise those compact Abelian groups having the property that every continuous function has a convergent Fourier series; and, in the remaining cases, applies the general method of Part 1 to construct continuous functions with divergent Fourier series.

Part 1: Boundedness principles

§ 1. *Introduction and preliminaries*

Let E denote a locally convex space and P a set of bounded gauges on E; that is, each $f \in P$ is a function with domain E and range a subset of $[0, \infty)$ such that

$$f(x+y) \leqq f(x) + f(y) \quad (x, y \in E),$$

$$f(\alpha x) = \alpha f(x) \quad (x \in E, \alpha > 0),$$

(so that $f(0) = 0$) and f is bounded on every bounded subset of E. In all cases, if f is continuous, then it is bounded; the converse is true if E is bornological ([2], p. 477). Note also that any seminorm is a positive gauge function; so too are $Re^+ u = \sup(Re\, u, 0)$ and $Im^+ u = \sup(Im\, u, 0)$, whenever u is a real-linear functional on E.

The boundedness principles discussed in this paper are those which assert that, granted suitable conditions on E, if the upper envelope f^* of P is finite valued, then f^* (which is evidently a gauge) is also bounded (cf. [2], Ch. 7).

It is customary to prove this type of boundedness principle (with continuous seminorms in place of bounded gauges) by appeal to assumed properties of E (for example, that it be second category, or barrelled, or sequentially complete and infrabarrelled) of a sort which renders the proof almost effortless.

One indirect use of boundedness principles aims at establishing the existence of misbehaviour, leaving aside any attempt to locate any specific instance thereof (cf. Banach's famous "principe de condensation des singularités"). We are here referring to situations in which a sequence (x_n) in E is known which satisfies

$$(x_n) \text{ is bounded (or convergent-to-zero) in } E \tag{1.1}$$

and

$$\sup_{n \in N} f^*(x_n) = \infty, \tag{1.2}$$

and an appeal to a boundedness principle is then made to infer the existence of one or more elements x of E satisfying

$$f^*(x) = \infty. \tag{1.3}$$

[The argument is simply that the negation of (1.3) implies, via a boundedness principle, that f^* is bounded (or continuous), and that this involves a contradiction of the conjunction of (1.1) and (1.2).]

The alternative to be advocated in this paper amounts to seeking a constructive procedure (involving no appeal to boundedness principles) leading from (1.1) and (1.2) to specified elements x satisfying (1.3). To do this seems all the more natural when, as is often the case, a fair amount of effort has already been expended in constructing a sequence (x_n) satisfying (1.1) and (1.2). Moreover, granted such a procedure, general boundedness principles can be derived quite easily (see §§ 3 and 4). This incidental approach to boundedness principles appears to be at least as successful as the customary one.

A construction of the desired type (a special case of which was subsequently located in the Appendix to [6]; see also [12], Solution 20 in [13], and [16]) is easily describable if E is complete and first countable (see § 2 below). The procedure is then extendible to sequentially complete spaces E (see § 3), and from this follows at once the corresponding version of the boundedness principle applying to bounded gauges (see § 4). Continuity of f^* follows under appropriate additional conditions.

Since we shall be working with gauge functions which are assumed to be merely bounded (rather than continuous), the usual standard passage from a non-Hausdorff space to its Hausdorff quotient is not generally available. For this reason, it seems worthwhile to formulate the results without assuming that E is Hausdorff. (If E is bornological—for example, first countable ([2], 6.1.1 and 7.3.2)—there is no problem.)

We shall write N for $\{1, 2, \ldots\}$; and the sequence $(u_n)_{n \in N}$ will often be written briefly as (u_n).

If E is any locally convex space and (x_n) a sequence of elements of E, the series $\sum_{n \in N} x_n$ or $\sum_{n=1}^{\infty} x_n$ is said to be *normally summable in* E if $\sum_{n \in N} \sigma(x_n) < \infty$ for every continuous seminorm σ on E. The series $\sum_{n \in N} x_n$ is said to be *convergent in* E and to have $x \in E$ as a sum, written $x \sim \sum_{n \in N} x_n$, if

$$\lim_{k \to \infty} \sigma(x - \sum_{n=1}^{k} x_n) = 0$$

for every continuous seminorm σ on E; the set of sums of a given convergent series form precisely one equivalence class modulo $\{0\}^-$. A series which is both normally summable and convergent in E is said to be *normally convergent in* E, or to *converge normally in* E. If E is sequentially complete, any series which is normally summable in E is normally convergent in E.

Two comments regarding the hypotheses imposed upon E are worth making at the outset. In the first place, we have concentrated on the locally convex case, with only Remarks 2.3 (3), 3.3 (3) and 4.2 (2) referring to the alternative, the reason being that this is by far the most important case for applications. Accordingly, throughout §§ 2-4, E *will* (except where the contrary is explicitly indicated) *be assumed to be locally convex.*

In the second place, it would suffice for subsequent developments to have Theorem 2.1 established for Banach spaces (and even merely for the familiar Banach space $l^1(N)$). However, only limited economy is gained by dealing with this special case alone and it seems best to retain a degree of generality which allows a more direct and explicit approach in the case of (say) Fréchet spaces.

Our final preliminary comment refers to boundedness of sets. If E is any topological linear space, a subset A of E will be said to be bounded in E if and only if to every neighbourhood U of 0 in E corresponds a number $r = r(A, U) > 0$ such that $rA = \{rx : x \in A\}$ is contained in U. If E is first countable and d is a semimetric on E defining its topology, boundedness in the above sense of a set $A \subseteq E$ must not be confused with metric boundedness [i.e., with the condition $\sup\{d(x, y) : x \in A, y \in A\} < \infty$]. It is in order to minimise the possibility of this confusion that we use the term "first countable" (an abbreviation for "satisfying the first axiom of countability") rather than "semimetrizable".

§ 2. *The construction when* E *is complete and first countable.*

In this section, where E will always denote a complete first countable (locally convex) space and P a set of bounded gauges on E, we will describe the basic construction. Let f^* denote the upper envelope of P.

If the sequence (x_n) figuring in (1.1) and (1.2) is such that $f^*(x_n) = \infty$ for some $n \in N$, no constructional problem remains. So we shall henceforth assume the contrary.

2.1 THEOREM. Suppose that β and α are real numbers satisfying $\beta > \alpha > 0$ and that sequences (x_n) in E, (f_n) in P are such that:

$$f^*(x_n) < \infty \quad \text{for every } n \in N, \tag{2.1}$$

$$\lim_{n \to \infty} x_n = 0, \tag{2.2}$$

$$\sup_{n \in N} f_n(x_n) = \infty. \tag{2.3}$$

Then infinite sequences $n_1 < n_2 < \ldots$ of positive integers may be constructed such that, for every sequence (γ_n) of real numbers satisfying

$$\alpha \leqq \gamma_n \leqq \beta \quad \text{for every } n \in N, \tag{2.4}$$

the series

$$\textstyle\sum_{\nu \in N} \gamma_\nu x_{n_\nu} \tag{2.5}$$

is normally convergent in E, and

$$f^*(x) \geqq \lim_{\nu \to \infty} f_{n_\nu}(x) = \infty \tag{2.6}$$

for each sum x of (2.5).

2.2 Construction and proof. Let (σ_ν) be an increasing sequence of continuous seminorms on E which define its topology. By initial passage to suitable subsequences, we may and will assume that (2.2) and (2.3) hold in the stronger form:

$$\sum_{n\in N} \sigma_n(x_n) < \infty, \tag{2.2'}$$

$$\lim_{n\to\infty} f_n(x_n) = \infty. \tag{2.3'}$$

[To do this, define $n_\nu \in N$ for $\nu \in N$ by induction in such a way that $n_1 < n_2 < \ldots$,

$$\sigma_\nu(x_{n_\nu}) \leqq 2^{-\nu} \quad \text{and} \quad f_{n_\nu}(x_{n_\nu}) > \nu \tag{2.7}$$

for all $\nu \in N$. This is possible since by (2.2) we can determine $\mathring{n}_1 \in N$ such that $\sigma_1(x_n) \leqq 2^{-1}$ if $n \geqq \mathring{n}_1$, and then, by (2.3) and the fact that each $f \in P$ is finite valued, there exists $n \geqq \mathring{n}_1$ such that $f_n(x_n) > 1$; denote the smallest such $n \geqq \mathring{n}_1$ by n_1. When $n_1 < n_2 < \ldots n_j$ have been determined so that (2.7) holds for $1 \leqq \nu \leqq j$, find (see (2.2)) an integer $\mathring{n}_{j+1} > n_j$ such that $\sigma_{j+1}(x_n) \leqq 2^{-j-1}$ if $n \geqq \mathring{n}_{j+1}$. Then (2.3) shows that there exists an integer $n \geqq \mathring{n}_{j+1}$ such that $f_n(x_n) > j+1$; put n_{j+1} for the smallest such integer $n \geqq \mathring{n}_{j+1}$.]

So now we assume (2.1), (2.2′) and (2.3′) and define one sequence $n_1 < n_2 < \ldots$ of the required type in the following manner. (Other possibilities are discussed in Remark 2.3 (2) below.) Let n_1 be the smallest $n \in N$ such that

$$f_n(x_n) \geqq \beta\alpha^{-1};$$

n_1 may be determined by (2.3′). Suppose that ν is a positive integer and that positive integers $n_1 < n_2 < \ldots < n_\nu$ have been defined so that

$$f_{n_j}(x_{n_\nu}) \leqq 2^{-\nu} \quad \text{whenever} \quad 1 \leqq j < \nu,$$

$$f_{n_\nu}(x_{n_\nu}) \geqq \beta\alpha^{-1} \textstyle\sum_{1\leqq j<\nu} f_{n_\nu}(x_{n_j}) + \beta\alpha^{-1}\nu.$$

[An empty sum is defined to be 0; then the conditions are all satisfied when $\nu = 1$.] Then (2.2′), (2.3′) and the fact that each $f \in P$ is finite-valued imply that there exists an integer $n > n_\nu$ which satisfies

$$f_{n_j}(x_n) \leqq 2^{-\nu-1} \quad \text{whenever} \quad 1 \leqq j < \nu+1,$$

$$f_n(x_n) \geqq \beta\alpha^{-1} \textstyle\sum_{1\leqq j<\nu+1} f_n(x_{n_j}) + \beta\alpha^{-1}(\nu+1);$$

let $n_{\nu+1}$ be the smallest such n. We then have for each $\nu \in N$:

$$n_\nu < n_{\nu+1},$$

$$f_{n_j}(x_{n_\nu}) \leqq 2^{-\nu} \quad \text{whenever} \quad 1 \leqq j < \nu, \tag{2.8}$$

$$f_{n_\nu}(x_{n_\nu}) \geqq \beta\alpha^{-1} \sum_{1 \leqq j < \nu} f_{n_\nu}(x_{n_j}) + \beta\alpha^{-1}\nu. \tag{2.9}$$

By (2.2′) and (2.4), the sum (2.5) is normally convergent in E. Let x be any sum of this series. To establish (2.6), write

$$x = u_\nu + \gamma_\nu x_{n_\nu} + v_\nu,$$

where $u_\nu = \sum_{1 \leqq j < \nu} \gamma_j x_{n_j}$ and v_ν is a sum of the series $\Sigma_{j > \nu} \gamma_j x_{n_j}$. Thus $\gamma_\nu x_{n_\nu} = x - u_\nu - v_\nu$, and so

$$\alpha f_{n_\nu}(x_{n_\nu}) \leqq f_{n_\nu}(\gamma_\nu x_{n_\nu}) \leqq f_{n_\nu}(x) + f_{n_\nu}(u_\nu) + f_{n_\nu}(v_\nu). \tag{2.10}$$

Now, by (2.4),

$$f_{n_\nu}(u_\nu) \leqq \beta \sum_{1 \leqq j < \nu} f_{n_\nu}(x_{n_j}); \tag{2.11}$$

and, by (2.4), (2.8) and the fact that each f_n is bounded, hence continuous,

$$f_{n_\nu}(v_\nu) \leqq \beta \sum_{j > \nu} f_{n_\nu}(x_{n_j}) \leqq \beta \sum_{j > \nu} 2^{-j} = \beta 2^{-\nu}. \tag{2.12}$$

By (2.10), (2.11) and (2.12)

$$\alpha f_{n_\nu}(x_{n_\nu}) \leqq f_{n_\nu}(x) + \beta \sum_{1 \leqq j < \nu} f_{n_\nu}(x_{n_j}) + \beta 2^{-\nu},$$

and so, by (2.9),

$$\beta \sum_{1 \leqq j < \nu} f_{n_\nu}(x_{n_j}) + \beta\nu \leqq f_{n_\nu}(x) + \beta \sum_{1 \leqq j < \nu} f_{n_\nu}(x_{n_j}) + \beta 2^{-\nu}.$$

Hence

$$f_{n_\nu}(x) \geqq \beta(\nu - 2^{-\nu}),$$

which proves (2.6) and the construction is complete.

2.3 REMARKS. (1) If it is known that

$$D = \{x \in E : f^*(x) < \infty\}$$

is dense in E, and if (x_n) and (f_n) satisfy (2.2) and (2.3), we can approximate each x_n so closely by an element y_n of D that (2.2) and (2.3) are left intact on replacing x_n by y_n. The hypotheses (2.1)—(2.3) are satisfied when x_n is everywhere replaced by y_n.

(2) If it be supposed that (2.2′) holds and that sequences (A_n), $(B_{n,r})$ and (C_n) are known such that $\lim_{n\to\infty} B_{n,r} = 0$ for every $r \in N$, $\lim_{n\to\infty} C_n = \infty$,

$$f^*(x_1) + \dots + f^*(x_n) \leqq A_n,$$

$$\max_{1\leqq j\leqq r} f_j(x_n) \leqq B_{n,r},$$

$$f_n(x_n) \geqq C_n,$$

then it is easy to specify a function $\phi_{\alpha,\beta} : N \times N \to N$ in terms of (A_n), $(B_{n,r})$ and (C_n) such that (2.4) and (2.5) yield (2.6) for every sequence (n_ν) such that $C_{n_1} \geqq \beta\alpha^{-1}$ and $n_{\nu+1} \geqq \phi_{\alpha,\beta}(n_\nu, \nu)$ for every $\nu \in N$.

(3) Local convexity of E is not essential in 2.1 and 2.2. In the contrary case one may proceed by introducing an invariant semimetric $(x, y) \mapsto |x-y|$ defining the topology of E, much as in [2], proof of Theorem 6.1.1, or [15], Chapitre I, § 3, No. 1. Normal summability in E of a series $\sum_{n\in N} z_n$ of elements of E may then be taken to mean the convergence of $\sum_{n\in N} |z_n|$. In place of (2.2′) arrange that

$$\sum_{n\in N} |\beta x_n| < \infty,$$

which will ensure the normal convergence in E of (2.5) whenever (2.4) holds (E being assumed to be complete). The rest of the proof and construction proceeds as before.

This method could, of course, be used when E is locally convex (and first countable and complete); we have not done so because the seminorms σ_n are usually more manageable in practice.

(4) A useful variant of 2.1 may be stated in the following terms.

2.4 Suppose given real numbers $\beta > \alpha > 0$ and sequences (x_n) in E and (f_n) in P such that

$$f^*(x_n) < \infty \quad \text{for every } n \in N, \tag{2.1}$$

$$\{x_n : n \in N\} \quad \text{is bounded in } E, \tag{2.2″}$$

$$\sup_{n\in N} f_n(x_n) = \infty. \tag{2.3}$$

Then one can construct a sequence (λ_n) of real numbers with the following properties:

$$\lambda_n \geqq 0, \sum_{n\in N} \lambda_n < \infty; \tag{2.13}$$

for every sequence (γ_n) satisfying (2.4) the series

$$\sum_{n\in N} \gamma_n \lambda_n x_n \tag{2.14}$$

is normally convergent in E; and

$$f^*(x) = \infty \tag{2.15}$$

for every sum x of the series (2.14).

In the sequel we shall denote by $l^1_+(N)$ the set of sequences (λ_n) satisfying (2.13).

PROOF. Define by recurrence a strictly increasing sequence (k_n) of positive integers, taking k_1 to the first $k \in N$ such that $f_k(x_k) > 1^3$ and k_{n+1} to be the first $k \in N$ such that $k > k_n$ and $f_k(x_k) > (n+1)^3$. Then apply 2.1 and 2.2 with x_n and f_n replaced by $n^{-2} x_{k_n}$ and f_{k_n} respectively. This furnishes at least one strictly increasing sequence (n_ν) of positive integers such that (2.4) entails that the series

$$\sum_{\nu\in N} \gamma_\nu \, n_\nu^{-2} \, x_{k_{n_\nu}} \tag{2.16}$$

is normally convergent in E and that (2.15) holds for every sum x of (2.16). It thus suffices to define λ_n to be n_ν^{-2} when $n = k_{n_\nu}$ for some $\nu \in N$ and to be zero for all other $n \in N$; it is obvious that (2.13) is then satisfied.

§ 3. *The construction when* E *is sequentially complete*

3.1 In this section we assume merely that E is a locally convex space which is sequentially complete. Again P will denote a set of bounded gauges on E, and f^* will denote its upper envelope. Suppose given sequences (x_n) in E and (f_n) in P such that (2.1), (2.2″) and (2.3) are satisfied. Then the conclusion of 2.4 remains valid.

PROOF. Consider the continuous linear map T of $l^1(N)$ into E defined by

$$T\xi = \sum_{n\in N} \xi_n \, x_n.$$

Evidently, $x_n = T\alpha_n$ for suitably chosen α_n such that $\{\alpha_n : n \in N\}$ is a bounded subset of $l^1(N)$. It therefore suffices to apply 2.4 with E replaced by $l^1(N)$, x_n by α_n, and f_n by $f_n \circ T$.

The following corollary will find application in §§ 5 and 6 below.

3.2 COROLLARY. Suppose that H is a Hausdorff topological linear space and that $(E_i)_{i\in I}$ is a family of linear subspaces of H such that

(i) E_i is a Banach space relative to a norm $\|\cdot\|_i$ and the injection $E_i \to H$ is continuous.

Let $\mathscr{E} = \bigcap \{E_i : i \in I\}$ be topologised as a topological linear space by taking a base at 0 in $\mathscr{E}$ formed of the sets $\{x \in \mathscr{E} : \sup_{i\in J} \| x \|_i < \varepsilon\}$, where ε ranges over positive numbers and J over finite subsets of I. Let E be a sequentially closed linear subspace of $\mathscr{E}$ and $(f_n)_{n\in N}$ a sequence of bounded gauges on E, and write f^* for the upper envelope of $(f_n)_{n\in N}$. Suppose finally that $(x_n)_{n\in N}$ is a sequence of elements of E such that

(ii) $f^*(x_n) < \infty$ for every $n \in N$;

(iii) $\sup_{n\in N} \| x_n \|_i < \infty$ for every $i \in I$;

(iv) $\sup_{n\in N} f_n(x_n) = \infty$.

The conclusion is that, given real numbers $\beta > \alpha > 0$, a sequence $(\lambda_n)_{n\in N} \in l^1_+(N)$ may be constructed such that, for every sequence $(\gamma_n)_{n\in N}$ satisfying (2.4), the series (2.14) is normally convergent in E to a (unique) sum x satisfying (2.15).

PROOF. In view of 3.1, it will suffice to verify that $\mathscr{E}$ (which is obviously locally convex) is sequentially complete and Hausdorff. The latter property is evidently present. As to the former, suppose that $(y_n)_{n\in N}$ is a Cauchy sequence in $\mathscr{E}$. Then, by definition of the topology on $\mathscr{E}$, (y_n) is Cauchy in E_i for every $i \in I$. Hence, by the first clause of (i), (y_n) is convergent in E_i to a limit $y_{(i)} \in E_i$. The second clause of (i), plus the fact that H is Hausdorff, entails that there exists $y \in H$ such that $y_{(i)} = y$ for every $i \in I$. Accordingly, $y \in \mathscr{E}$; and, since $\lim_{n\to\infty} y_n = y_{(i)} = y$ in E_i for every $i \in I$, $\lim_{n\to\infty} y_n = y$ in $\mathscr{E}$. This shows that $\mathscr{E}$ is sequentially complete.

3.3 REMARKS. (1) If the elements of P are seminorms (rather than merely gauges), we may everywhere permit (γ_n) to be a sequence taking values in the (real or complex) scalar field of E, replacing (2.4) by the condition

$$\alpha \leqq | \gamma_n | \leqq \beta \quad \text{for every } n \in N. \tag{2.4'}$$

This is easily seen by reverting to 2.2 and using the fact that now $f_n(\gamma x) = | \gamma | f_n(x)$ for every $x \in E$, every $n \in N$ and every scalar γ. No changes are needed in the choice of the n_ν.

(2) Local convexity is needed in the proof of 3.1 since otherwise (2.2″), i.e., the boundedness of $S = \{x_n : n \in N\}$ in E, does not guarantee the existence of any continuous or bounded linear map T from $l^1(N)$ into E such that S is contained in the T-image of a bounded subset of $l^1(N)$. For it is plain that such a T can exist, only if the convex envelope S' of S is bounded in E. On the other hand, it is not difficult to verify that any first countable linear topological space E, in which the convex envelope of every bounded set (or of the range of every sequence converging to zero in E) is bounded, is necessarily locally convex.

(3) Naturally, local convexity of E may be dropped from the hypotheses of 3.1, if one assumes in place of (2.2″) that the convex envelope of $\{x_n : n \in N\}$ is a bounded subset of E.

§ 4. *Deduction of boundedness principles*

4.1 THEOREM. Suppose that E is a sequentially complete locally convex space and that P is a set of bounded gauges on E. If $f^*(x) = \sup \{f(x) : f \in P\} < \infty$ for every $x \in E$, then f^* is bounded.

PROOF. Suppose the contrary, that is, that $f^*(x) < \infty$ for every $x \in E$ and yet there exists a bounded subset B of E on which f^* is unbounded. Then we can choose $x_n \in B$, $f_n \in P$ such that $f_n(x_n) > n$ for every $n \in N$. Then (2.1), (2.2″) and (2.3) are satisfied; hence, by 3.1, there exists $x \in E$ such that $f^*(x) = \infty$, which is the required contradiction.

4.2 REMARKS. (1) If we assume also that E is infrabarrelled and that each $f \in P$ is continuous, it follows that f^* is continuous, that is, that P is equicontinuous if it is pointwise bounded; cf. [2], pp. 47, 480-81. For, if V denotes the interval $[-\varepsilon, \varepsilon]$, where $\varepsilon > 0$, then

$$f^{*-1}(V) = \bigcap \{f^{-1}(V) : f \in P\}$$

is closed, convex and balanced and absorbs bounded sets in E. Since E is infrabarrelled, $f^{*-1}(V)$ is therefore a neighbourhood of the origin in E and thus f^* is continuous, as asserted.

(2) If one drops the hypothesis that E be locally convex (the remaining assumptions of Theorem 4.1 remaining intact), the substance of Remark 3.3 (3) shows that one may still conclude that $f^*(B)$ is bounded whenever B is a subset of E whose convex envelope in E is bounded.

However, even assuming that E is first countable and complete, one can in general no longer conclude that f^* is bounded (i.e., that $f^*(A)$ is bounded for every bounded subset A of E) whenever it is finite-valued. Counter-examples are easily given in the case of the familiar spaces $E = l^p(N)$ with $p \in (0, 1)$.

PART 2: APPLICATIONS TO MULTIPLIERS

§ 5. (p, q)-*multipliers which are not measures*

5.1 INTRODUCTION. In this section and the following one we will use the substance of § 3 to prove several apparently new properties of (p, q)-multipliers. Let G be a locally compact group [all topological groups will be assumed to be Hausdorff and, in this section, will be multiplicatively written with identity e]. Denote by $L^p(G)$, where $1 \leqq p \leqq \infty$, the usual Lebesgue space formed with a fixed left Haar measure λ_G on G; and by $C_c(G)$ the space of continuous complex-valued functions on G with compact supports.

For $a \in G$, define the left translation operator τ_a and the right translation operator ρ_a by

$$\tau_a g(x) = g(a^{-1} x) \quad \text{and} \quad \rho_a g(x) = g(xa^{-1});$$

respectively. A linear operator T from $C_c(G)$ into $L^q(G)$ is said to be a *(left) (p, q)-multiplier* if and only if

(i) T is continuous from $C_c(G)$, equipped with the norm induced by $L^p(G)$, into $L^q(G)$; and

(ii) T commutes with left translations, that is $T\tau_a = \tau_a T$ for all $a \in G$.

A *right (p, q)-multiplier* is defined in a similar manner with (ii) replaced by

(ii′) $T\rho_a = \rho_a T$ for all $a \in G$.

Let $L_p^q(G)$ denote the Banach space of (p, q)-multipliers equipped with the customary norm, denoted by $\| \cdot \|_{p,q}$, of continuous linear operators from a subspace of $L^p(G)$ into $L^q(G)$. That is, for each $T \in L_p^q(G)$, $\| T \|_{p,q}$ is the smallest real number K satisfying

$$\| Tg \|_q \leqq K \| g \|_p$$