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A NAIVELY CONSTRUCTIVE APPROACH TO BOUNDEDNESS PRINCIPLES, WITH APPLICATIONS TO HARMONIC ANALYSIS

by R. E. Edwards and J. F. Price

General Introduction

This paper is partly pedagogical and expository. Thus Part 1 (§§ 1-4) presents a naively constructive approach to boundedness principles. Although this construction leads to results differing but slightly from the standard versions, we feel that this approach (which can be followed with no overt reference to category, barrelled spaces, and so on) offers some pedagogical and expository advantages. We emphasise that the level of constructivity is naive and not fundamental.

The remainder of the paper consists of applications of the constructive procedure. In Part 2 (§§ 5, 6) the applications yield improvements of recent results due to Price and to Gaudry concerning multipliers. In Part 3 (§§ 7-10) the applications are to convergence and divergence of Fourier series of continuous functions on compact Abelian groups. These results (which may be known to the afficionados but which, as far as we know, have not been published hitherto) characterise those compact Abelian groups having the property that every continuous function has a convergent Fourier series; and, in the remaining cases, applies the general method of Part 1 to construct continuous functions with divergent Fourier series.

Part 1: Boundedness principles

§ 1. *Introduction and preliminaries*

Let E denote a locally convex space and P a set of bounded gauges on E; that is, each $f \in P$ is a function with domain E and range a subset of $[0, \infty)$ such that

$$f(x+y) \leqq f(x) + f(y) \quad (x, y \in E),$$

$$f(\alpha x) = \alpha f(x) \quad (x \in E, \alpha > 0),$$

(so that $f(0) = 0$) and f is bounded on every bounded subset of E. In all cases, if f is continuous, then it is bounded; the converse is true if E is bornological ([2], p. 477). Note also that any seminorm is a positive gauge function; so too are $Re^+ u = \sup(Re\, u, 0)$ and $Im^+ u = \sup(Im\, u, 0)$, whenever u is a real-linear functional on E.

The boundedness principles discussed in this paper are those which assert that, granted suitable conditions on E, if the upper envelope f^* of P is finite valued, then f^* (which is evidently a gauge) is also bounded (cf. [2], Ch. 7).

It is customary to prove this type of boundedness principle (with continuous seminorms in place of bounded gauges) by appeal to assumed properties of E (for example, that it be second category, or barrelled, or sequentially complete and infrabarrelled) of a sort which renders the proof almost effortless.

One indirect use of boundedness principles aims at establishing the existence of misbehaviour, leaving aside any attempt to locate any specific instance thereof (cf. Banach's famous "principe de condensation des singularités"). We are here referring to situations in which a sequence (x_n) in E is known which satisfies

$$(x_n) \text{ is bounded (or convergent-to-zero) in } E \tag{1.1}$$

and

$$\sup_{n \in N} f^*(x_n) = \infty, \tag{1.2}$$

and an appeal to a boundedness principle is then made to infer the existence of one or more elements x of E satisfying

$$f^*(x) = \infty. \tag{1.3}$$

[The argument is simply that the negation of (1.3) implies, via a boundedness principle, that f^* is bounded (or continuous), and that this involves a contradiction of the conjunction of (1.1) and (1.2).]

The alternative to be advocated in this paper amounts to seeking a constructive procedure (involving no appeal to boundedness principles) leading from (1.1) and (1.2) to specified elements x satisfying (1.3). To do this seems all the more natural when, as is often the case, a fair amount of effort has already been expended in constructing a sequence (x_n) satisfying (1.1) and (1.2). Moreover, granted such a procedure, general boundedness principles can be derived quite easily (see §§ 3 and 4). This incidental approach to boundedness principles appears to be at least as successful as the customary one.

A construction of the desired type (a special case of which was subsequently located in the Appendix to [6]; see also [12], Solution 20 in [13], and [16]) is easily describable if E is complete and first countable (see § 2 below). The procedure is then extendible to sequentially complete spaces E (see § 3), and from this follows at once the corresponding version of the boundedness principle applying to bounded gauges (see § 4). Continuity of f^* follows under appropriate additional conditions.

Since we shall be working with gauge functions which are assumed to be merely bounded (rather than continuous), the usual standard passage from a non-Hausdorff space to its Hausdorff quotient is not generally available. For this reason, it seems worthwhile to formulate the results without assuming that E is Hausdorff. (If E is bornological—for example, first countable ([2], 6.1.1 and 7.3.2)—there is no problem.)

We shall write N for $\{1, 2, ...\}$; and the sequence $(u_n)_{n \in N}$ will often be written briefly as (u_n).

If E is any locally convex space and (x_n) a sequence of elements of E, the series $\sum_{n \in N} x_n$ or $\sum_{n=1}^{\infty} x_n$ is said to be *normally summable in* E if $\sum_{n \in N} \sigma(x_n) < \infty$ for every continuous seminorm σ on E. The series $\sum_{n \in N} x_n$ is said to be *convergent in* E and to have $x \in E$ as a sum, written $x \sim \sum_{n \in N} x_n$, if

$$\lim_{k \to \infty} \sigma(x - \textstyle\sum_{n=1}^{k} x_n) = 0$$

for every continuous seminorm σ on E; the set of sums of a given convergent series form precisely one equivalence class modulo $\{0\}^-$. A series which is both normally summable and convergent in E is said to be *normally convergent in E*, or to *converge normally in E*. If E is sequentially complete, any series which is normally summable in E is normally convergent in E.

Two comments regarding the hypotheses imposed upon E are worth making at the outset. In the first place, we have concentrated on the locally convex case, with only Remarks 2.3 (3), 3.3 (3) and 4.2 (2) referring to the alternative, the reason being that this is by far the most important case for applications. Accordingly, throughout §§ 2-4, E *will* (except where the contrary is explicitly indicated) *be assumed to be locally convex.*

In the second place, it would suffice for subsequent developments to have Theorem 2.1 established for Banach spaces (and even merely for the familiar Banach space $l^1(N)$). However, only limited economy is gained by dealing with this special case alone and it seems best to retain a degree of generality which allows a more direct and explicit approach in the case of (say) Fréchet spaces.

Our final preliminary comment refers to boundedness of sets. If E is any topological linear space, a subset A of E will be said to be bounded in E if and only if to every neighbourhood U of 0 in E corresponds a number $r = r(A, U) > 0$ such that $rA = \{rx : x \in A\}$ is contained in U. If E is first countable and d is a semimetric on E defining its topology, boundedness in the above sense of a set $A \subseteq E$ must not be confused with metric boundedness [i.e., with the condition $\sup\{d(x, y) : x \in A, y \in A\} < \infty$]. It is in order to minimise the possibility of this confusion that we use the term "first countable" (an abbreviation for "satisfying the first axiom of countability") rather than "semimetrizable".

§ 2. *The construction when* E *is complete and first countable.*

In this section, where E will always denote a complete first countable (locally convex) space and P a set of bounded gauges on E, we will describe the basic construction. Let f^* denote the upper envelope of P.

If the sequence (x_n) figuring in (1.1) and (1.2) is such that $f^*(x_n) = \infty$ for some $n \in N$, no constructional problem remains. So we shall henceforth assume the contrary.

2.1 Theorem. Suppose that β and α are real numbers satisfying $\beta > \alpha > 0$ and that sequences (x_n) in E, (f_n) in P are such that:

$$f^*(x_n) < \infty \quad \text{for every } n \in N, \tag{2.1}$$

$$\lim_{n \to \infty} x_n = 0, \tag{2.2}$$

$$\sup_{n \in N} f_n(x_n) = \infty. \tag{2.3}$$

Then infinite sequences $n_1 < n_2 < \ldots$ of positive integers may be constructed such that, for every sequence (γ_n) of real numbers satisfying

$$\alpha \leqq \gamma_n \leqq \beta \quad \text{for every } n \in N, \tag{2.4}$$

the series

$$\sum_{\nu \in N} \gamma_\nu x_{n_\nu} \tag{2.5}$$

is normally convergent in E, and

$$f^*(x) \geqq \lim_{\nu \to \infty} f_{n_\nu}(x) = \infty \tag{2.6}$$

for each sum x of (2.5).

2.2 Construction and proof. Let (σ_ν) be an increasing sequence of continuous seminorms on E which define its topology. By initial passage to suitable subsequences, we may and will assume that (2.2) and (2.3) hold in the stronger form:

$$\sum_{n \in N} \sigma_n(x_n) < \infty, \tag{2.2'}$$

$$\lim_{n \to \infty} f_n(x_n) = \infty. \tag{2.3'}$$

[To do this, define $n_\nu \in N$ for $\nu \in N$ by induction in such a way that $n_1 < n_2 < \dots$,

$$\sigma_\nu(x_{n_\nu}) \leqq 2^{-\nu} \quad \text{and} \quad f_{n_\nu}(x_{n_\nu}) > \nu \tag{2.7}$$

for all $\nu \in N$. This is possible since by (2.2) we can determine $\mathring{n}_1 \in N$ such that $\sigma_1(x_n) \leqq 2^{-1}$ if $n \geqq \mathring{n}_1$, and then, by (2.3) and the fact that each $f \in P$ is finite valued, there exists $n \geqq \mathring{n}_1$ such that $f_n(x_n) > 1$; denote the smallest such $n \geqq \mathring{n}_1$ by n_1. When $n_1 < n_2 < \dots n_j$ have been determined so that (2.7) holds for $1 \leqq \nu \leqq j$, find (see (2.2)) an integer $\mathring{n}_{j+1} > n_j$ such that $\sigma_{j+1}(x_n) \leqq 2^{-j-1}$ if $n \geqq \mathring{n}_{j+1}$. Then (2.3) shows that there exists an integer $n \geqq \mathring{n}_{j+1}$ such that $f_n(x_n) > j+1$; put n_{j+1} for the smallest such integer $n \geqq \mathring{n}_{j+1}$.]

So now we assume (2.1), (2.2′) and (2.3′) and define one sequence $n_1 < n_2 < \dots$ of the required type in the following manner. (Other possibilities are discussed in Remark 2.3 (2) below.) Let n_1 be the smallest $n \in N$ such that

$$f_n(x_n) \geqq \beta\alpha^{-1};$$

n_1 may be determined by (2.3′). Suppose that ν is a positive integer and that positive integers $n_1 < n_2 < \dots < n_\nu$ have been defined so that

$$f_{n_j}(x_{n_\nu}) \leqq 2^{-\nu} \quad \text{whenever} \quad 1 \leqq j < \nu,$$

$$f_{n_\nu}(x_{n_\nu}) \geqq \beta\alpha^{-1} \sum_{1 \leqq j < \nu} f_{n_\nu}(x_{n_j}) + \beta\alpha^{-1}\nu.$$

[An empty sum is defined to be 0; then the conditions are all satisfied when $\nu = 1$.] Then (2.2′), (2.3′) and the fact that each $f \in P$ is finite-valued imply that there exists an integer $n > n_\nu$ which satisfies

$$f_{n_j}(x_n) \leqq 2^{-\nu-1} \quad \text{whenever} \quad 1 \leqq j < \nu + 1,$$

$$f_n(x_n) \geqq \beta\alpha^{-1} \sum_{1 \leqq j < \nu+1} f_n(x_{n_j}) + \beta\alpha^{-1}(\nu+1);$$

let $n_{\nu+1}$ be the smallest such n. We then have for each $\nu \in N$:

$$n_\nu < n_{\nu+1},$$

$$f_{n_j}(x_{n_\nu}) \leqq 2^{-\nu} \quad \text{whenever} \quad 1 \leqq j < \nu, \tag{2.8}$$

$$f_{n_\nu}(x_{n_\nu}) \geqq \beta\alpha^{-1} \textstyle\sum_{1 \leqq j < \nu} f_{n_\nu}(x_{n_j}) + \beta\alpha^{-1}\nu. \tag{2.9}$$

By (2.2′) and (2.4), the sum (2.5) is normally convergent in E. Let x be any sum of this series. To establish (2.6), write

$$x = u_\nu + \gamma_\nu x_{n_\nu} + v_\nu,$$

where $u_\nu = \sum_{1 \leqq j < \nu} \gamma_j x_{n_j}$ and v_ν is a sum of the series $\Sigma_{j>\nu} \gamma_j x_{n_j}$. Thus $\gamma_\nu x_{n_\nu} = x - u_\nu - v_\nu$, and so

$$\alpha f_{n_\nu}(x_{n_\nu}) \leqq f_{n_\nu}(\gamma_\nu x_{n_\nu}) \leqq f_{n_\nu}(x) + f_{n_\nu}(u_\nu) + f_{n_\nu}(v_\nu). \tag{2.10}$$

Now, by (2.4),

$$f_{n_\nu}(u_\nu) \leqq \beta \textstyle\sum_{1 \leqq j < \nu} f_{n_\nu}(x_{n_j}); \tag{2.11}$$

and, by (2.4), (2.8) and the fact that each f_n is bounded, hence continuous,

$$f_{n_\nu}(v_\nu) \leqq \beta \textstyle\sum_{j>\nu} f_{n_\nu}(x_{n_j}) \leqq \beta \sum_{j>\nu} 2^{-j} = \beta 2^{-\nu}. \tag{2.12}$$

By (2.10), (2.11) and (2.12)

$$\alpha f_{n_\nu}(x_{n_\nu}) \leqq f_{n_\nu}(x) + \beta \textstyle\sum_{1 \leqq j < \nu} f_{n_\nu}(x_{n_j}) + \beta 2^{-\nu},$$

and so, by (2.9),

$$\beta \textstyle\sum_{1 \leqq j < \nu} f_{n_\nu}(x_{n_j}) + \beta\nu \leqq f_{n_\nu}(x) + \beta \sum_{1 \leqq j < \nu} f_{n_\nu}(x_{n_j}) + \beta 2^{-\nu}.$$

Hence

$$f_{n_\nu}(x) \geqq \beta(\nu - 2^{-\nu}),$$

which proves (2.6) and the construction is complete.

2.3 Remarks. (1) If it is known that

$$D = \{x \in E : f^*(x) < \infty\}$$

is dense in E, and if (x_n) and (f_n) satisfy (2.2) and (2.3), we can approximate each x_n so closely by an element y_n of D that (2.2) and (2.3) are left intact on replacing x_n by y_n. The hypotheses (2.1)—(2.3) are satisfied when x_n is everywhere replaced by y_n.

(2) If it be supposed that (2.2′) holds and that sequences (A_n), $(B_{n,r})$ and (C_n) are known such that $\lim_{n\to\infty} B_{n,r} = 0$ for every $r \in N$, $\lim_{n\to\infty} C_n = \infty$,

$$f^*(x_1) + \dots + f^*(x_n) \leqq A_n,$$

$$\max_{1\leqq j\leqq r} f_j(x_n) \leqq B_{n,r},$$

$$f_n(x_n) \geqq C_n,$$

then it is easy to specify a function $\phi_{\alpha,\beta} : N \times N \to N$ in terms of (A_n), $(B_{n,r})$ and (C_n) such that (2.4) and (2.5) yield (2.6) for every sequence (n_ν) such that $C_{n_1} \geqq \beta\alpha^{-1}$ and $n_{\nu+1} \geqq \phi_{\alpha,\beta}(n_\nu, \nu)$ for every $\nu \in N$.

(3) Local convexity of E is not essential in 2.1 and 2.2. In the contrary case one may proceed by introducing an invariant semimetric $(x, y) \mapsto |x-y|$ defining the topology of E, much as in [2], proof of Theorem 6.1.1, or [15], Chapitre I, § 3, No. 1. Normal summability in E of a series $\sum_{n\in N} z_n$ of elements of E may then be taken to mean the convergence of $\sum_{n\in N} |z_n|$. In place of (2.2′) arrange that

$$\sum_{n\in N} |\beta x_n| < \infty,$$

which will ensure the normal convergence in E of (2.5) whenever (2.4) holds (E being assumed to be complete). The rest of the proof and construction proceeds as before.

This method could, of course, be used when E is locally convex (and first countable and complete); we have not done so because the seminorms σ_n are usually more manageable in practice.

(4) A useful variant of 2.1 may be stated in the following terms.

2.4 Suppose given real numbers $\beta > \alpha > 0$ and sequences (x_n) in E and (f_n) in P such that

$$f^*(x_n) < \infty \quad \text{for every } n \in N, \tag{2.1}$$

$$\{x_n : n \in N\} \quad \text{is bounded in } E, \tag{2.2''}$$

$$\sup_{n\in N} f_n(x_n) = \infty. \tag{2.3}$$

Then one can construct a sequence (λ_n) of real numbers with the following properties:

$$\lambda_n \geqq 0, \sum_{n\in N} \lambda_n < \infty; \tag{2.13}$$

for every sequence (γ_n) satisfying (2.4) the series

$$\sum_{n \in N} \gamma_n \lambda_n x_n \tag{2.14}$$

is normally convergent in E; and

$$f^*(x) = \infty \tag{2.15}$$

for every sum x of the series (2.14).

In the sequel we shall denote by $l^1_+(N)$ the set of sequences (λ_n) satisfying (2.13).

PROOF. Define by recurrence a strictly increasing sequence (k_n) of positive integers, taking k_1 to the first $k \in N$ such that $f_k(x_k) > 1^3$ and k_{n+1} to be the first $k \in N$ such that $k > k_n$ and $f_k(x_k) > (n+1)^3$. Then apply 2.1 and 2.2 with x_n and f_n replaced by $n^{-2} x_{k_n}$ and f_{k_n} respectively. This furnishes at least one strictly increasing sequence (n_ν) of positive integers such that (2.4) entails that the series

$$\sum_{\nu \in N} \gamma_\nu \, n_\nu^{-2} \, x_{k_{n_\nu}} \tag{2.16}$$

is normally convergent in E and that (2.15) holds for every sum x of (2.16). It thus suffices to define λ_n to be n_ν^{-2} when $n = k_{n_\nu}$ for some $\nu \in N$ and to be zero for all other $n \in N$; it is obvious that (2.13) is then satisfied.

§ 3. *The construction when* E *is sequentially complete*

3.1 In this section we assume merely that E is a locally convex space which is sequentially complete. Again P will denote a set of bounded gauges on E, and f^* will denote its upper envelope. Suppose given sequences (x_n) in E and (f_n) in P such that (2.1), (2.2″) and (2.3) are satisfied. Then the conclusion of 2.4 remains valid.

PROOF. Consider the continuous linear map T of $l^1(N)$ into E defined by

$$T\xi = \sum_{n \in N} \xi_n \, x_n.$$

Evidently, $x_n = T\alpha_n$ for suitably chosen α_n such that $\{\alpha_n : n \in N\}$ is a bounded subset of $l^1(N)$. It therefore suffices to apply 2.4 with E replaced by $l^1(N)$, x_n by α_n, and f_n by $f_n \circ T$.

The following corollary will find application in §§ 5 and 6 below.

3.2 COROLLARY. Suppose that H is a Hausdorff topological linear space and that $(E_i)_{i \in I}$ is a family of linear subspaces of H such that

(i) E_i is a Banach space relative to a norm $\|\cdot\|_i$ and the injection $E_i \to H$ is continuous.

Let $\mathscr{E} = \bigcap \{E_i : i \in I\}$ be topologised as a topological linear space by taking a base at 0 in $\mathscr{E}$ formed of the sets $\{x \in \mathscr{E} : \sup_{i \in J} \| x \|_i < \varepsilon\}$, where ε ranges over positive numbers and J over finite subsets of I. Let E be a sequentially closed linear subspace of $\mathscr{E}$ and $(f_n)_{n \in N}$ a sequence of bounded gauges on E, and write f^* for the upper envelope of $(f_n)_{n \in N}$. Suppose finally that $(x_n)_{n \in N}$ is a sequence of elements of E such that

(ii) $f^*(x_n) < \infty$ for every $n \in N$;

(iii) $\sup_{n \in N} \| x_n \|_i < \infty$ for every $i \in I$;

(iv) $\sup_{n \in N} f_n(x_n) = \infty$.

The conclusion is that, given real numbers $\beta > \alpha > 0$, a sequence $(\lambda_n)_{n \in N} \in l^1_+(N)$ may be constructed such that, for every sequence $(\gamma_n)_{n \in N}$ satisfying (2.4), the series (2.14) is normally convergent in E to a (unique) sum x satisfying (2.15).

PROOF. In view of 3.1, it will suffice to verify that $\mathscr{E}$ (which is obviously locally convex) is sequentially complete and Hausdorff. The latter property is evidently present. As to the former, suppose that $(y_n)_{n \in N}$ is a Cauchy sequence in $\mathscr{E}$. Then, by definition of the topology on $\mathscr{E}$, (y_n) is Cauchy in E_i for every $i \in I$. Hence, by the first clause of (i), (y_n) is convergent in E_i to a limit $y_{(i)} \in E_i$. The second clause of (i), plus the fact that H is Hausdorff, entails that there exists $y \in H$ such that $y_{(i)} = y$ for every $i \in I$. Accordingly, $y \in \mathscr{E}$; and, since $\lim_{n \to \infty} y_n = y_{(i)} = y$ in E_i for every $i \in I$, $\lim_{n \to \infty} y_n = y$ in $\mathscr{E}$. This shows that $\mathscr{E}$ is sequentially complete.

3.3 REMARKS. (1) If the elements of P are seminorms (rather than merely gauges), we may everywhere permit (γ_n) to be a sequence taking values in the (real or complex) scalar field of E, replacing (2.4) by the condition

$$\alpha \leqq |\gamma_n| \leqq \beta \quad \text{for every } n \in N. \tag{2.4'}$$

This is easily seen by reverting to 2.2 and using the fact that now $f_n(\gamma x) = |\gamma| f_n(x)$ for every $x \in E$, every $n \in N$ and every scalar γ. No changes are needed in the choice of the n_ν.

(2) Local convexity is needed in the proof of 3.1 since otherwise (2.2″), i.e., the boundedness of $S = \{x_n : n \in N\}$ in E, does not guarantee the existence of any continuous or bounded linear map T from $l^1(N)$ into E such that S is contained in the T-image of a bounded subset of $l^1(N)$. For it is plain that such a T can exist, only if the convex envelope S' of S is bounded in E. On the other hand, it is not difficult to verify that any first countable linear topological space E, in which the convex envelope of every bounded set (or of the range of every sequence converging to zero in E) is bounded, is necessarily locally convex.

(3) Naturally, local convexity of E may be dropped from the hypotheses of 3.1, if one assumes in place of (2.2″) that the convex envelope of $\{x_n : n \in N\}$ is a bounded subset of E.

§ 4. *Deduction of boundedness principles*

4.1 THEOREM. Suppose that E is a sequentially complete locally convex space and that P is a set of bounded gauges on E. If $f^*(x) = \sup \{f(x) : f \in P\} < \infty$ for every $x \in E$, then f^* is bounded.

PROOF. Suppose the contrary, that is, that $f^*(x) < \infty$ for every $x \in E$ and yet there exists a bounded subset B of E on which f^* is unbounded. Then we can choose $x_n \in B$, $f_n \in P$ such that $f_n(x_n) > n$ for every $n \in N$. Then (2.1), (2.2″) and (2.3) are satisfied; hence, by 3.1, there exists $x \in E$ such that $f^*(x) = \infty$, which is the required contradiction.

4.2 REMARKS. (1) If we assume also that E is infrabarrelled and that each $f \in P$ is continuous, it follows that f^* is continuous, that is, that P is equicontinuous if it is pointwise bounded; cf. [2], pp. 47, 480-81. For, if V denotes the interval $[-\varepsilon, \varepsilon]$, where $\varepsilon > 0$, then

$$f^{*-1}(V) = \bigcap \{f^{-1}(V) : f \in P\}$$

is closed, convex and balanced and absorbs bounded sets in E. Since E is infrabarrelled, $f^{*-1}(V)$ is therefore a neighbourhood of the origin in E and thus f^* is continuous, as asserted.

(2) If one drops the hypothesis that E be locally convex (the remaining assumptions of Theorem 4.1 remaining intact), the substance of Remark 3.3 (3) shows that one may still conclude that $f^*(B)$ is bounded whenever B is a subset of E whose convex envelope in E is bounded.

However, even assuming that E is first countable and complete, one can in general no longer conclude that f^* is bounded (i.e., that $f^*(A)$ is bounded for every bounded subset A of E) whenever it is finite-valued. Counter-examples are easily given in the case of the familiar spaces $E = l^p(N)$ with $p \in (0, 1)$.

PART 2: APPLICATIONS TO MULTIPLIERS

§ 5. (p, q)-*multipliers which are not measures*

5.1 INTRODUCTION. In this section and the following one we will use the substance of § 3 to prove several apparently new properties of (p, q)-multipliers. Let G be a locally compact group [all topological groups will be assumed to be Hausdorff and, in this section, will be multiplicatively written with identity e]. Denote by $L^p(G)$, where $1 \leqq p \leqq \infty$, the usual Lebesgue space formed with a fixed left Haar measure λ_G on G; and by $C_c(G)$ the space of continuous complex-valued functions on G with compact supports.

For $a \in G$, define the left translation operator τ_a and the right translation operator ρ_a by

$$\tau_a g(x) = g(a^{-1} x) \quad \text{and} \quad \rho_a g(x) = g(xa^{-1});$$

respectively. A linear operator T from $C_c(G)$ into $L^q(G)$ is said to be a *(left)* (p, q)*-multiplier* if and only if

(i) T is continuous from $C_c(G)$, equipped with the norm induced by $L^p(G)$, into $L^q(G)$; and

(ii) T commutes with left translations, that is $T\tau_a = \tau_a T$ for all $a \in G$.

A *right* (p, q)*-multiplier* is defined in a similar manner with (ii) replaced by

(ii′) $T\rho_a = \rho_a T$ for all $a \in G$.

Let $L_p^q(G)$ denote the Banach space of (p, q)-multipliers equipped with the customary norm, denoted by $\|\cdot\|_{p,q}$, of continuous linear operators from a subspace of $L^p(G)$ into $L^q(G)$. That is, for each $T \in L_p^q(G)$, $\| T \|_{p,q}$ is the smallest real number K satisfying

$$\| Tg \|_q \leqq K \| g \|_p$$

for all $g \in C_c(G)$. [When $p \neq \infty$ it is more usual to define $L_p^q(G)$ as the space of unique continuous extensions to $L^p(G)$ of the (p, q)-multipliers.]

As an example, whenever $k \in C_c(G)$, the operator T_k, defined by

$$T_k : g \mapsto g * k$$

for all $g \in C_c(G)$, is (a) a (p, q)-multiplier for all (p, q) satisfying $1 \leqq p \leqq q \leqq \infty$; and (b) a (p, q)-multiplier for all $p, q \in [1, \infty]$ provided G is compact. [When G is noncompact it is known that $L_p^q = \{0\}$ whenever $p > q$—see [1], § 3.4.3. We also remark that, unless a more explicit reference is given, all the properties of the convolution operator between functions and functions and between functions and measures used in the sequel may be found in [2], § 4.19.] For convenience, we will sometimes write $\| k \|_{p,q}$ in place of $\| T_k \|_{p,q}$. Use will be made of the fact that

$$\left.\begin{aligned} \| k \|_{1,s} &= \| T_k \|_{1,s} = \| k \|_s, \\ \| k \|_{s,\infty} &= \| T_k \|_{s,\infty} = \| \Delta^{-1/s'} k \|_{s'}, \end{aligned}\right\} \tag{5.1}$$

where Δ denotes the modular function of G, as defined in [7], (15.11) and (15.15) and s' is defined by $1/s + 1/s' = 1$; cf. [1], Corollary 2.6.2 (i) and Theorem 1.4.

5.2 Definitions. If $T \in L_p^q(G)$, we say that:

(i) supp $T \subseteq W$, where W is a closed subset of G, if and only if supp $Tg \subseteq (\operatorname{supp} g) \,.\, W$ for every $g \in C_c(G)$.

(ii) T *is a measure* μ if and only if $Tg = g * \mu$ for every $g \in C_c(G)$.

[When $k \in C_c(G)$, supp $T_k \subseteq W$ if and only if supp $k \subseteq W$; and in any case T_k is the measure $\mu = k\lambda_G$.]

5.3 Adjoint multipliers. Let $T \in L_p^q(G)$ and define an adjoint T' of T by

$$g * T' h\,(e) = Tg * h\,(e) \tag{5.2}$$

for all $g, h \in C_c(G)$. Since $Tg * h(e) = \int_G Tg \,.\, \check{h}\, d\lambda_G$, where $\check{h}(x) = h(x^{-1})$, it is readily shown that T' commutes with right translations and that it may be extended to an operator from $(L^{q'})^\vee$ into $(L^{p'})^\vee$. We also infer from (5.2) that

$$g * T' h = Tg * h \tag{5.3}$$

everywhere on G, since $\tau_a(Tg * h) = \tau_a(Tg) * h = T(\tau_a g) * h$. It is plain from (5.3) that T is a measure μ if and only if T' is of the form $h \mapsto \mu * h$.

If we also assume that G is unimodular, so that the L^p norms of g and $\check{g}$ are identical, two applications of the converse to Hölder's inequality will show that

$$\| T' \|_{q',p'} = \| T \|_{p,q}, \tag{5.4}$$

where $1/p' + 1/p = 1$; thus T' is a right (q', p')-multiplier. Moreover (cf. [1], Corollary 2.6.2 (ii))

$$\| T'_k \|_{1,s} = \| k \|_{1,s} = \| k \|_s. \tag{5.5}$$

5.4 Rudin-Shapiro sequences. If U is a nonvoid open subset of G, by a *U-supported Rudin-Shapiro sequence* (briefly: a *U-RS-sequence*) on G we shall mean a sequence $(h_n)_{n \in N}$ of elements of $C_c(G)$ with the following properties:

$$\left.\begin{aligned} &\operatorname{supp} h_n \subseteq U, \\ &\inf \| h_n \|_2 > 0, \quad \sup \| h_n \|_\infty < \infty, \\ &\lim_{n \to \infty} \| h_n \|_{2,2} = 0. \end{aligned}\right\} \tag{5.6}$$

We do not know conditions on G which are necessary and sufficient for there to exist U-RS-sequences on G for a given U. When G is nondiscrete Abelian, U-RS-sequences may be constructed on G in a fairly explicit manner for every non-void open subset U of G (see Appendix A.2 below). Sufficient conditions applying in the non-Abelian case are given in Appendix A.3.

If (h_n) is a U-RS-sequence, we may construct positive integers $m_1 < m_2 < \ldots$ so that

$$\| h_{m_n} \|_{2,2} \leqq n^{-1} 2^{-n}.$$

Let $k_n = n h_{m_n}$. It then follows from (5.6) that

$$\| k_n \|_1 \geqq Bn, \tag{5.7}$$

$$\| k_n \|_s \leqq A^{1/s} n \quad (1 \leqq s \leqq \infty), \tag{5.8}$$

$$\| k_n \|_{2,2} \leqq 2^{-n}, \tag{5.9}$$

where A and B are positive and independent of n.

5.5 When G is infinite compact Abelian, Theorem 4.15 of [1] shows that there exists an operator belonging to $L_p^q(G)$ for every $p \in (1, \infty]$ and every $q \in [1, \infty)$ and which is not a measure. [Given an infinite Sidon subset of Γ, operators with this property are immediately constructible whether G is Abelian or not; cf. [7], (37.22).] When G is noncompact locally compact Abelian or infinite compact, it has recently been shown that there exists an operator belonging to $L_p^p(G)$ for every $p \in (1, \infty)$ which is not a bounded measure. [See [4] and [9]; the proof contained in [9] is constructive to some extent. See also [17].] We aim to show in 5.7 below that, if U is a relatively compact open subset of G, and if we are able to construct a U-RS-sequence on G, then we can construct an operator $T \in \bigcap \{L_p^q(G) : 1 < p \leqq q \leqq \infty\}$ such that supp $T \subseteq \bar{U}$ and T is not a measure. (If G is also unimodular, an analogous result holds for right (p, q)-multipliers.)

The inequality $p > 1$, along with the inequality $q < \infty$ if G is unimodular, is essential for the existence of such a T since every member of $L_1^q(G)$ is of the form $g \mapsto g * \mu$, where μ is a bounded measure if $q = 1$ or $\mu \in L^q(G)$ if $1 < q \leqq \infty$ (see [1], Corollary 2.6.2), and since $L_1^q(G) = L_{q'}^\infty(G)$ if G is unimodular (see (5.4) above). When G is non-compact, the inequality $p \leqq q$ is also essential since in this case $L_p^q(G) = \{0\}$ whenever $p > q$ (see [1], § 3.4.3). Concerning non-unimodular groups, see 5.8 below.

5.6 Lemma. Let k be a continuous function supported by a relatively compact open subset U of G, and let $c = c(U) > 0$ denote $\inf\{\Delta(x)^{-1} : x \in U\}$, where Δ is the modular function for G. Then functions $u, v \in C_c(G)$ with $\| u * v \|_\infty \leqq 1$ may be constructed so that

$$| u * T_k v(e) | \geqq (c/2) \| k \|_1.$$

Proof. Let $\{\delta_\alpha\}$ be an approximate identity on G comprised of non-negative functions with compact supports and which each satisfy $\int_G \delta_\alpha d\lambda_G = 1$. Since $\check{k} * \delta_\alpha$ tends to $\check{k}$ in $L^1(G)$, we may select $v = \check{\delta}_\alpha$ so that

$$\| (v*k)^\vee \|_1 = \| \check{k} * \check{v} \|_1 \geqq \frac{3}{4} \| \check{k} \|_1. \tag{5.10}$$

Define a compactly supported function g on G by $g(x) = \overline{v * k(x)}/|v * k(x)|$ if $v * k(x) \neq 0$, and $g(x) = 0$ otherwise. Let $u_\alpha = \delta_\alpha * \check{g}$. Then $u_\alpha \in C_c(G)$ and, since $u_\alpha (v * k)^\vee$ tends to $\check{g} (v * k)^\vee$ in $L^1(G)$, we may select α so that

$$\left| \int_G u_\alpha (v * k)^\vee \, d\lambda_G \right| \geqq \frac{3}{4} \left| \int \check{g} \, (v * k)^\vee \, d\lambda_G \right|. \tag{5.11}$$

Putting $u = u_\alpha$, we then have from (5.10) and (5.11)

$$\begin{aligned} \left| u * T_k \, v(e) \right| &= \left| \int_G u \, (v * k)^\vee \, d\lambda_G \right| \\ &\geqq \frac{3}{4} \left| \int_G \check{g} \, (v * k)^\vee \, d\lambda_G \right| \\ &= \frac{3}{4} \| (v * k)^\vee \|_1 \geqq \frac{1}{2} \| \check{k} \|_1 \\ &\geqq (c/2) \| k \|_1. \end{aligned}$$

Moreover, $\| u * v \|_\infty = \| \check{v} * \check{u} \|_\infty \leqq \| \check{v} \|_1 \| \check{u} \|_\infty \leqq 1$, as required.

5.7 Theorem. (1) Let (h_n) be a U-RS-sequence on a locally compact group G, where U is a relatively compact open subset of G, and let $(k_n)_{n \in N}$ be defined as in 5.4. A continuum of sequences $(\omega_n) \in l^1_+(N)$ may be constructed for which the series

$$\sum_{n \in N} \omega_n \, T_{k_n} \tag{5.12}$$

converges normally in $L^q_p(G)$ for every pair (p, q) satisfying $1 < p \leqq q < \infty$ to a unique operator, T say, such that

(i) supp $T \subseteq \overline{U}$, and

(ii) T is not a measure.

(2) With the further condition that G is unimodular, the theorem remains valid if we replace throughout left multipliers and their related concepts by right multipliers and their correspondingly related concepts.

Proof. (1) For each $n \in N$, Lemma 5.6 shows that we may select and fix $u_n, v_n \in C_c(G)$ such that

$$\| u_n * v_n \|_\infty \leqq 1 \, , \left| u_n * T_{k_n} v_n (e) \right| \geqq (c/2) \| k_n \|_1, \tag{5.13}$$

where $c = \inf \{ \Delta(x)^{-1} : x \in U \} > 0$ does not depend on n.

We aim to apply 3.2, taking:

H = the space of linear maps from $C_c(G)$ into $L^1_{loc}(G)$, the topology on H being that of pointwise convergence;

$I = \{(p, q) : 1 < p \leqq q < \infty\};$

$E_{(p,q)} = L_p^q(G)$ with its standard norm;

$E = \mathscr{E};$

$f_n : T \mapsto | u_n * Tv_n(e) |;$

$x_n = T_{k_n}.$

It is clear that 3.2 (i) holds and that f_n is continuous (a fortiori bounded) on E. By way of verification of 3.2 (ii)-(iv) we will show that

$$f^*(T_{k_n}) < \infty \text{ for every } n \in N, \tag{5.14}$$

$$\lim_{n\to\infty} T_{k_n} = 0 \text{ in } E, \tag{5.15}$$

$$\lim_{n\to\infty} f_n(T_{k_n}) = \infty. \tag{5.16}$$

Regarding (5.14), we have

$$f_m(T_{k_n}) = | u_m * T_{k_n} v_m(e) | = | u_m * v_m * k_n(e) | \leqq \| u_m * v_m \|_\infty \| \check{k}_n \|_1$$

which, by the first clause of (5.13), does not exceed $\| \check{k}_n \|_1$. Hence $f^*(T_{k_n}) \leqq \| \check{k} \|_1$, which is finite since $k_n \in C_c(G)$.

As to (5.15), the Riesz-Thorin convexity theorem ([11], Volume II, p. 95) shows that for $(p, q) \in I$ satisfying $\frac{1}{p} + \frac{1}{q} \geqq 1$ one has

$$\| T_{k_n} \|_{p,q} \leqq \| T_{k_n} \|_{2,2}^{\alpha} \| T_{k_n} \|_{1,s}^{1-\alpha}, \tag{5.17}$$

where $1/p = \alpha/2 + (1-\alpha)/1$, $1/q = \alpha/2 + (1-\alpha)/s$, so that $\alpha = 2/p' \in (0, 1]$ and $s \in [1, \infty]$. On combining the first clause of (5.1), (5.8), (5.9) and (5.17), we see that

$$\lim_{n\to\infty} \| T_{k_n} \|_{p,q} = 0 \tag{5.18}$$

for every pair $(p, q) \in I$ satisfying $1/p + 1/q \geqq 1$. If, on the other hand, $(p, q) \in I$ and $1/p + 1/q < 1$, a similar argument gives

$$\| T_{k_n} \|_{p,q} \leqq \| T_{k_n} \|_{2,2}^{\alpha} \| T_{k_n} \|_{s,\infty}^{1-\alpha} \tag{5.19}$$

where $1/p = \alpha/2 + (1-\alpha)/s$ and $1/q = \alpha/2$, so that $\alpha = 2/q \in (0, 1)$ and $s \in (1, \infty]$. On combining the second clause of (5.1), (5.8), (5.9) and the fact that Δ is bounded away from zero on U, (5.18) appears once more. The verification of (5.15) is thus complete.

The definition of f_n combines with (5.7) and (5.13) to yield (5.16).

Appeal to 3.2 provides a construction for a continuum of sequences $(\omega_n) \in l^1_+(N)$ for each of which the series (5.12) converges normally in E to a sum T satisfying

$$f^*(T) = \infty. \tag{5.20}$$

This entails that, for every $(p, q) \in I$, $T \in L^q_p(G)$ and the series (5.12) is normally convergent in $L^q_p(G)$ to the sum T. Since supp $T_{k_n} \subseteq U$ for every n, it is simple to verify that sup $T \subseteq \bar{U}$. It remains to show that T is not a measure. However, were T to be the measure μ, it would be the case that supp $\mu \subseteq \bar{U}$ and so, using the first clause of (5.7), that

$$\begin{aligned} f_n(T) &= | u_n * Tv_n(e) | = | u_n * v_n * \mu(e) | \\ &= | \textstyle\int_G (u_n * v_n)^\vee \Delta^{-1} d\mu | \\ &\leqq \textstyle\int_G \Delta^{-1} d|\mu|. \end{aligned}$$

Since μ has a compact support, this inequality would lead to a contradiction of (5.20). Thus T cannot be a measure.

(2) Finally, when G is unimodular, everything remains valid when right multipliers replace left multipliers throughout: this can be seen by either repeating the entire argument ab initio, or by deriving it from the result already obtained by making use of the properties of the adjoint discussed in 5.3.

5.8 The non-unimodular case. (i) If G is non-unimodular, there can be no full analogue of Theorem 5.7 applying to right multipliers. This is so because in this case there exist no non-trivial right (p, q)-multipliers when $p \neq q$.

To see this, suppose that T is a right (p, q)-multiplier and that $p \neq q$. For $f \in C_c(G)$ and $a \in G$ we then have

$$\| \rho_a Tf \|_q = \| T\rho_a f \|_q \leqq \| T \|_{p,q} \| \rho_a f \|_p = \| T \|_{p,q} \Delta(a)^{1/p} \| f \|_p$$

and

$$\| \rho_a Tf \|_q = \Delta(a)^{1/q} \| Tf \|_q.$$

Hence

$$\| Tf \|_q \leqq \Delta(a)^{1/p - 1/q} \| T \|_{p,q} \| f \|_p.$$

Since G is non-unimodular and $p \neq q$,

$$\inf_{a \in G} \Delta(a)^{1/p - 1/q} = 0,$$

and we infer that $T = 0$.

(ii) In spite of (i) immediately above, there is a partial analogue taking the following form.

Assume that there exists a sequence (h_n) satisfying (5.6), where now $\| h_n \|_{2,2}$ is defined to mean

$$\sup \{ \| h_n * f \|_2 : f \in C_c(G) , \| f \|_2 \leqq 1 \}.$$

Then modification of the proof of Theorem 5.7 will lead to the construction of operators T which are right multipliers of type (p, p) for every $p \in (1, \infty)$, have supports contained in $\bar{U}$, and are not of the form $f \mid\rightarrow \mu * f$ for any measure μ.

§ 6. (p, q)-*multipliers whose transforms are not measures*

6.1 Introduction. Throughout this section we suppose that G is a locally compact Abelian (= LCA) group with dual group Γ, both groups being additively written. We begin by slightly modifying the form of the definition of (p, q)-multipliers, so rendering it possible to make certain statements about their Fourier transforms without attempting a general definition of such transforms. To this end, let F denote the set of functions on G which belong to $\bigcap \{L^p(G) : 1 \leqq p \leqq \infty\}$ and which possess Fourier transforms with compact supports, and denote by $L_p^q(G)$ the set of continuous linear operators from F, equipped with the $L^p(G)$-norm, into $L^q(G)$ which commute with translations. As before, equip $L_p^q(G)$ with the $(L^p(G), L^q(G))$ operator norm. It is easy to specify a natural isometry between $L_p^q(G)$ as defined above and $L_p^q(G)$ as defined in § 5, and so we speak of the elements of $L_p^q(G)$ as (p, q)-multipliers on G.

When T is a (p, q)-multiplier in this sense, we say that its *Fourier transform* $\hat{T}$ *is a measure* μ if and only if there exists a measure μ on Γ such that

$$h * Tg\,(0) = \int_\Gamma \hat{h}\,\hat{g}\,d\mu \tag{6.1}$$

for all $g, h \in F$, where $\hat{u}$ denotes the Fourier transform of u. Similarly, if Ω is an open subset of Γ, we shall write $\hat{T} = \mu$ on Ω if and only if (6.1) holds for all g, $h \in F$ such that supp $\hat{g} \subseteq \Omega$. If Σ is a closed subset of Γ, we shall write supp $\hat{T} \subseteq \Sigma$ if and only if $\hat{T} = 0$ on Γ/Σ.

It is simple to verify that, if $K \in F$ and T_K is the mapping $g \mapsto g * K = K * g$, then $T_K \in L_p^q$ whenever $1 \leqq p \leqq q \leqq \infty$. (In fact, $\| K * g \|_\infty \leqq \| K \|_{p'} \| g \|_p$ and $\| K * g \|_p \leqq \| K \|_1 \| g \|_p$ and the convexity of the function $t \mapsto \log \| K * g \|_{t^{-1}}$, or an appeal to the closed graph theorem, does the rest.) Furthermore, $\hat{T}_K$ is the measure $\hat{K}\lambda_\Gamma$, where λ_Γ is the Haar measure of Γ normalised so that the $L^2(\lambda_\Gamma)$-norm of $\hat{u}$ is equal to $\| u \|_2$ for every $u \in L^2(G)$.

6.2 It has been shown by Gaudry ([5], Theorem 3.1) that, if G is noncompact LCA and $1 \leqq p < 2 < q \leqq \infty$, there exist operators $T \in L_p^q(G)$ such that $\hat{T}$ is not a measure. In 6.3 and its proof we shall indicate how to construct operators T which belong to $L_p^q(G)$ for every pair (p, q) satisfying $1 \leqq p < 2 < q \leqq \infty$ and which are such that supp $\hat{T}$ is contained in a compact subset of Γ and $\hat{T}$ is not a measure. The precise statement of 6.3 requires some prefatory remarks.

Let G be a noncompact LCA group and Ω a relatively compact open subset of the dual group Γ. Since Γ is nondiscrete LCA, an Ω-RS-sequence (h_n) on Γ may be constructed in such a way that the inverse Fourier transform of h_n belongs to $L^1(G)$ for every n; see Appendix A.2. Assuming this to have been done, choose positive integers $m_1 < m_2 < \ldots$ and define $k_n = nh_{m_n}$ exactly as in 5.4, so that (5.7)-(5.9) remain intact (but with Γ, rather than G, as the underlying group). We now consider the functions K_n on G, K_n being defined to be the inverse Fourier transform of k_n.

It is plain that every K_n belongs to F. Moreover, an application of Hölder's inequality yields

$$\| K_n \|_s \leqq \| K_n \|_2^{2/s} \| K_n \|_\infty^{1-2/s} \quad (s > 2). \tag{6.2}$$

By Parseval's formula and (5.8),

$$\| K_n \|_2 = \| k_n \|_2 \leqq A^{\frac{1}{2}} n;$$

also, since G is LCA, (5.9) leads to

$$\| K_n \|_\infty = \| T_{k_n} \|_{2,2} \leqq 2^{-n}.$$

Inserting these last two estimates into (6.2), we obtain

$$\| K_n \|_s = 0\,(n^{2/s}\, 2^{-n(1-2/s)}) \quad (s > 2). \tag{6.3}$$

We shall need to note also that a construction, similar to that appearing in the proof of Lemma 5.6, shows that for each $n \in N$ we may select and fix $u_n, v_n \in F$ such that

$$\| \hat{u}_n \hat{v}_n \|_\infty \leqq 1 \tag{6.4}$$

and

$$\left| \int_\Gamma \hat{u}_n \hat{v}_n \hat{K}_n \, d\lambda_\Gamma \right| \geqq \tfrac{1}{2} \| \hat{K}_n \|_1 = \tfrac{1}{2} \| k_n \|_1 \geqq \tfrac{1}{2} Bn, \tag{6.5}$$

the last link in this chain of inequalities stemming from (5.7).

6.3 Theorem. Let G be a noncompact LCA group, Ω a relatively compact open subset of the dual group Γ. Suppose the function $K_n (n \in N)$ to be defined as in 6.2. A continuum of sequences $(\omega_n) \in l^1_+(N)$ may be constructed, for each of which the series

$$\textstyle\sum_{n \in N} \omega_n T_{K_n} \tag{6.6}$$

converges normally in $L^q_p(G)$ for every pair (p, q) satisfying $1 \leqq p < 2 < q \leqq \infty$, the sum T of the series (6.6) satisfying the conditions

(i) $T \in \bigcap \{ L^q_p(G) : 1 \leqq p < 2 < q \leqq \infty \}$;

(ii) supp $\hat{T} \subseteq \Omega$; and

(iii) $\hat{T}$ is not a measure.

Proof. Since G is Abelian, (5.4) shows that $L^q_p(G) = L^{p'}_{q'}(G)$ and $\| \cdot \|_{p,q} = \| \cdot \|_{q',p'}$. Accordingly, we may and will restrict attention to those pairs (p, q) such that $1 \leqq p < 2 < q \leqq \infty$ and $1/p + 1/q \geqq 1$; denote by I the set of such pairs.

We propose to appeal to Corollary 3.2, taking therein

$H =$ the space of linear maps from F into $L^1_{loc}(G)$ with the topology of pointwise convergence;

I as defined immediately above;

$E_{(p,q)} = L^q_p(G)$ for every $(p, q) \in I$;

$E =$ the closed linear subspace of $\mathscr{E}$ generated by the T_{K_n} $(n \in N)$;

$f_n : T \mapsto | u_n * Tv_n(0) |$;

$x_n = T_{K_n}$.

Regarding the hypotheses of Corollary 3.2, it is clear that 3.2 (i) is satisfied. Also, for any $T \in E$ and any $m \in N$, Hölder's inequality yields

$$f_m(T) \leqq \| u_m \|_{q'} \| T v_m \|_q \leqq \| u_m \|_{q'} \| T \|_{p,q} \| v_m \|_{p},$$

which, since u_m and v_m belong to F, shows that f_m is continuous (and therefore certainly bounded) on E.

Next, since (see the remarks at the end of 6.1 above) $\hat{T}_{K_n}$ is the measure $\hat{K}_n \lambda_\Gamma = k_n \lambda_\Gamma$,

$$f_m(T_{K_n}) = | \int_\Gamma \hat{u}_m \, \hat{v}_m \, k_n \, d\lambda_\Gamma | \leqq \| k_n \|_1,$$

the inequality coming from (6.4). This makes it clear that $f^* (T_{K_n})$ is finite for every $n \in N$, so that 3.2 (ii) is satisfied.

Turning to 3.2 (iii), note first that by convexity (as in the proof of (5.17)) we have

$$\| T_{K_n} \|_{p,q} \leqq \| T_{K_n} \|_{2,2}^{\alpha} \| T_{K_n} \|_{1,s}^{1-\alpha}, \tag{6.7}$$

where, since $p < 2 < q$, we have $\alpha < 1$ and $s > 2$. Now, by the case $s = \infty$ of (5.8),

$$\| T_{K_n} \|_{2,2} = \| \hat{K}_n \|_\infty = \| k_n \|_\infty \leqq n.$$

Using this in combination with (6.3) and (6.7), it appears that

$$\| T_{K_n} \|_{p,q} = 0 \, (n^\alpha n^{2(1-\alpha)/s} \, 2^{-\beta n}),$$

where $\beta = (1-\alpha)(1-2/s)$ is positive, and so

$$\lim_{n \to \infty} T_{K_n} = 0 \text{ in } E,$$

which is more than enough to verify 3.2 (iii).

As for 3.2 (iv), the fact that $\hat{T}_{K_n} = \hat{K}_n \lambda_\Gamma$ combines with (6.5) to yield

$$f_n (T_{K_n}) = | \int_\Gamma \hat{u}_n \, \hat{v}_n \, \hat{K}_n \, d\lambda_\Gamma | \geqq \tfrac{1}{2} B n,$$

which confirms 3.2 (iv).

An appeal to Corollary 3.2 is thus justified and assures one of the existence of a continuum of sequences $(\omega_n) \in l^1_+(N)$ for each of which the series (6.6) converges normally to a (unique) sum T in E which satisfies

$$f^* (T) = \infty. \tag{6.8}$$

From this it is evident that (i) is satisfied, and that, for every pair (p, q)

satisfying $1 \leqq p < 2 < q \leqq \infty$, the series (6.6) converges normally in $L_p^q(G)$ to T. Next, T is the limit in E of

$$S_r = \sum_{n=1}^{r} \omega_n T_{K_n}$$

as $r \to \infty$ and, since it is plain that supp $S_r \subseteq \Omega$ for every r, (ii) is easily derived. Finally, if $\hat{T}$ were a measure μ, it would necessarily be the case that supp $\mu \subseteq \bar{\Omega}$ and so, for every $n \in N$, one would have by (6.1) and (6.4)

$$\begin{aligned} f_n(T) &= | u_n * Tv_n(0) | = | \textstyle\int_\Gamma \hat{u}_n \hat{v}_n \, d\mu | \\ &\leqq | \mu | (\bar{\Omega}), \end{aligned}$$

which is finite since Ω is relatively compact. However, this plainly would entail $f^*(T) < \infty$, in conflict with (6.8), so that T cannot be a measure and (iii) is verified. This completes the proof.

6.4 Remark. Theorem 6.3 was proved by Hörmander ([14], Theorem 1.9) for $G = R^n$ and any given pair (p, q) satisfying $1 \leqq p < 2 < q \leqq \infty$, this result being extended to a general noncompact LCA G by Gaudry [5]. The argument given by Hörmander (loc. cit. Theorem 1.6 and the remark immediately following) for the case $G = R^n$ can also be extended to a general LCA G and shows that, if *either* $q \leqq 2$ *or* $p \geqq 2$, then every $T \in L_p^q(G)$ is such that $\hat{T}$ is a measure [and indeed a measure of the form $\psi\lambda_\Gamma$, where $\psi \in L_{loc}^2(\Gamma)$ if $q \leqq 2$ and $\psi \in L_{loc}^p(\Gamma)$ if $p \geqq 2$, and so $\psi \in L_{loc}^2(\Gamma)$ in either case]. Thus the hypotheses made in Theorem 6.3 about p and q are necessary for the validity of the conclusion.

Part 3: Applications to Fourier series

§ 7. *Applications to divergence of Fourier series.*

7.1 Throughout §§ 7-10, G will denote an infinite Hausdorff compact Abelian group with character group Γ, and λ_G the Haar measure on G, normalised so that $\lambda_G(G) = 1$. For any $f \in L^1(G)$, $\hat{f}$ will denote the Fourier transform of f; for any finite subset Δ of Γ,

$$S_\Delta f = \sum_{\gamma \in \Delta} \hat{f}(\gamma)\gamma \tag{7.1}$$

is the Δ-partial sum of the Fourier series of f; and sp (f) will stand for the spectrum of f, i.e., for the support supp $\hat{f} = \{\gamma \in \Gamma : \hat{f}(\gamma) \neq 0\}$ of $\hat{f}$. The term "trigonometric polynomial" will frequently be abbreviated to "t.p.". In addition, Φ will denote the largest torsion subgroup of Γ ([7], (A.4)), and π the natural map of Γ onto Γ/Φ. If Δ denotes a subset of Γ, $[\Delta]$ will stand for the subgroup of Γ generated by Δ.

By a *(convergence) grouping* we shall mean a sequence $\mathscr{D} = (\Delta_j)_{j \in N} = (\Delta_j)$ of finite subsets Δ_j of Γ such that

$$\left.\begin{array}{l} \Delta_j \subseteq \Delta_{j+1} \quad (j \in N); \\ \bigcup_{j=1}^{\infty} \Delta_j = \Gamma_0 \text{ is a subgroup of } \Gamma \text{, said to be} \\ \qquad \textit{covered by } \mathscr{D}; \\ \text{for each } j \in N,\ \Delta_j = \Omega_j + \Lambda_j \text{, where } \Lambda_j \text{ is a} \\ \qquad \text{nonvoid finite subset of } \Phi \text{ and } \Omega_j \text{ is a finite} \\ \qquad \text{subset of } \Gamma \text{ such that } \pi \mid \Omega_j \text{ is 1-1.} \end{array}\right\} \tag{7.2}$$

[The first two conditions are natural enough in the context described in 7.3, but the third is less so and may well be pointless.] The grouping $\mathscr{D}$ is said to be of *infinite type* if and only if $\pi(\Gamma_0)$ is infinite.

7.2 EXAMPLES. (i) Let Γ_0 be any countable subgroup of Γ such that $\Gamma_0 \cap \Phi = \{0\}$; for example, $\Gamma_0 = \{n\gamma_0 : n \in Z\}$, where $\gamma_0 \in \Gamma \setminus \Phi$. Then a grouping $\mathscr{D}$ covering Γ_0 results whenever $\Lambda_j = \{0\}$ and $\Delta_j = \Omega_j$ for every $j \in N$, where $(\Omega_j)_{j \in N}$ is any increasing sequence of finite subsets of Γ_0 with union equal to Γ_0. This grouping is of infinite type if and only if Γ_0 is infinite.

(ii) If G is connected, and if Γ_0 is any countable subgroup of Γ, then ([10], 2.5.6 (c), 8.1.2 (a) and (b) and 8.1.6) Γ_0 is an ordered group isomorphic to a discrete subgroup of R. Assuming $\Gamma_0 \neq \{0\}$, Γ_0 has a smallest positive element γ_0 and $\Gamma_0 = \{n\gamma_0 : n \in Z\}$. A natural grouping $\mathscr{D}$ covering Γ_0 is that in which $\Lambda_j = \{0\}$ and

$$\Delta_j = \Omega_j = \{n\gamma_0 : n \in Z, |n| \leq j\}$$

for every $j \in N$; this grouping is of infinite type.

7.3 A grouping $\mathscr{D} = (\Delta_j)_{j \in N}$ will be thought of as specifying one of the many possible ways in which one may interpret the convergence of Fourier series of functions f on G satisfying $sp(f) \subseteq \Gamma_0$, namely, as convergence of the corresponding sequence of partial sums $(S_{\Delta_j} f)_{j \in N}$.

Indeed, the conditions (7.2) guarantee that $\lim_{j\to\infty} S_{\Delta_j} f = f$ for all sufficiently regular such functions f. However, our concern rests with the possibility of constructing continuous functions f on G satisfying

$$\operatorname{sp}(f) \subseteq \Gamma_0, \ \overline{\lim_{j\to\infty}} \operatorname{Re} S_{\Delta_j} f(0) = \infty. \tag{7.3}$$

It will appear that the possibilities exhibit a fairly clear dichotomy, depending largely upon whether G is or is not 0-dimensional.

In the first place, it will emerge in 7.6 that the construction principle of § 2, applied to the Banach space $E = C(G)$ of continuous complex valued functions on G [with norm $\|\cdot\|$ equal to the maximum modulus] and to sequences of gauges of the type

$$f \mapsto \operatorname{Re} S_\Delta f(0) = \operatorname{Re} \int_G D_\Delta f \, d\lambda_G, \tag{7.4}$$

where D_Δ stands for the "Dirichlet function"

$$D_\Delta = \sum_{\gamma\in\Delta} \bar{\gamma}, \tag{7.5}$$

shows that the problem hinges on the existence of groupings $\mathscr{D}$ for which

$$\rho_j = \| D_{\Delta_j} \|_1 = \int_G | D_{\Delta_j} | \, d\lambda_G \to \infty. \tag{7.6}$$

Accordingly, and in view of the fact ([7], (24.26)) that G is 0-dimensional if and only if Γ coincides with Φ, it emerges that the dichotomy referred to may be expressed in the following way.

7.4 Two cases arise, namely:

(i) G is not 0-dimensional (i.e., $\Phi \neq \Gamma$). Then (see Example 7.2 (i)) there exist groupings $\mathscr{D} = (\Delta_j)$ of infinite type; and, for any such grouping, one can construct (fairly explicitly, as described in 7.6) continuous functions f on G satisfying (7.3). In particular [cf. Example 7.2 (i)], if Γ_0 is any countably infinite subgroup of Γ satisfying $\Gamma_0 \cap \Phi = \{0\}$, and if $(\Delta_j)_{j\in N}$ is any increasing sequence of finite subsets of Γ_0 with union Γ_0, we can construct a continuous f on G satisfying (7.3).

(ii) G is 0-dimensional (i.e., $\Phi = \Gamma$). Then there exists no grouping of infinite type. However, given any countable subgroup Γ_0 of Γ, there are groupings $\mathscr{D} = (\Delta_j)$ covering Γ_0, in which $\Omega_j = \{0\}$ and $\Delta_j = \Lambda_j$ is a finite subgroup of Γ_0, and for which

$$f = \lim_{j\to\infty} S_{\Delta_j} f$$

uniformly on G for every continuous f satisfying $\mathrm{sp}(f) \subseteq \Gamma_0$.

Case (i) will be dealt with in § 8, case (ii) in § 9. The groupings described in case (ii) prove to be exceptional in various ways; see 9.3.

7.5 REMARK. Perhaps it should be stressed here that, if Γ_0 is any infinite subgroup of Γ, there is no obstacle to constructing continuous functions f such that $\mathrm{sp}(f) \subseteq \Gamma_0$ and finite subsets $\Delta_j \subseteq \Delta_{j+1}$ of Γ_0 for which

$$\lim_j S_{\Delta_j} f(0) = \infty.$$

[One has in fact only to construct a continuous f such that $\mathrm{sp}(f) \subseteq \Gamma_0$ and $\sum_{\gamma \in \Gamma} | \hat{f}(\gamma) | = \infty$; it is then trivial that there exist finite subsets Δ of Γ_0 for which $| S_\Delta f(0) |$ is arbitrarily large, so that we can choose a sequence (Δ_j) for which $\Delta_j \subseteq \Delta_{j+1}$ and $| S_{\Delta_j} f(0) | \to \infty$ with j.] However, the sets Δ_j obtained this way will not [and, in view of 7.4 (ii), cannot] in general be such that $\bigcup_{j=1}^{\infty} \Delta_j = \Gamma_0$. For more details, see A.5.1 and A.5.2 of the Appendix.

7.6 Suppose one is given a grouping $\mathscr{D} = (\Delta_j)_{j \in N}$ covering Γ_0 and satisfying (7.6). As is described in § 10, one may construct polynomials $q_{p_j, \nu}$ in two indeterminates over the real field (ν being a suitable fixed integer not less than 36 and p_j any positive number not less than $\| D_{\Delta_j} \|_\infty$) such that, for suitable unimodular complex numbers ξ_j, the t.p.s

$$Q_j = \xi_j \left(1 + \frac{1}{\nu}\right)^{-1} q_{p_j, \nu} (D_{\Delta_j}, \bar{D}_{\Delta_j})$$

satisfy

$$\left.\begin{aligned} &\| Q_j \| \leqq 1,\ sp(Q_j) \subseteq [\Delta_j] \subseteq \Gamma_0, \\ &S_{\Delta_j} Q_j(0) = \textstyle\int_G D_{\Delta_j} Q_j \, d\lambda_G \text{ is real and } \geqq \tfrac{1}{2} \rho_j. \end{aligned}\right\} \tag{7.7}$$

In view of (7.2), (7.6) and (7.7), one may choose inductively a sequence $(j_n)_{n \in N}$ of positive integers so that

$$\left.\begin{aligned} &S_{\Delta_{j_n}} Q_{j_n}(0) \text{ is real and } > n^3, \\ &j_n < j_{n+1},\ sp(Q_{j_n}) \subseteq \Gamma_0. \end{aligned}\right\} \tag{7.8}$$

Accordingly, the t.p.s

$$u_n = n^{-2}\, Q_{j_n}$$

satisfy the conditions

$$\left.\begin{aligned} &\operatorname{sp}(u_n) \subseteq \Gamma_0,\ \sum_{n=1}^{\infty} \| u_n \| < \infty \\ &S_{\Delta_{j_n}} u_n(0) \text{ is real and } > n. \end{aligned}\right\} \tag{7.9}$$

At this point the construction in § 2 will yield integers $0 < n_1 < n_2 < \ldots$ and specifiable sequences $(\gamma_p)_{p \in N}$ of positive numbers such that each function of the form

$$f = \sum_{p=1}^{\infty} \gamma_p\, u_{n_p}$$

is continuous and satisfies

$$sp(f) \subseteq \Gamma_0,\ \lim_{p \to \infty} \operatorname{Re} S_{\Delta_{j_{n_p}}} f(0) = \infty. \tag{7.10}$$

A fortiori, f satisfies (7.3).

We add here that, if the Δ_j are symmetric, the D_{Δ_j} are real-valued, and we may work throughout with real-valued functions, replacing $\operatorname{Re} S_{\Delta_j} f$ by $S_{\Delta_j} f$ everywhere.

§ 8. *Discussion of case (i) :* G *not* 0-*dimensional*

8.1 In this case $\Phi \neq \Gamma$, and we begin by considering a finite subset of Γ of the form

$$\Delta = \Omega + \Lambda, \tag{8.1}$$

where Ω and Λ are finite subsets of Γ such that $\pi \mid \Omega$ is 1-1 and $\varnothing \neq \Lambda \subseteq \Phi$. We aim to show that (for a suitable absolute constant $k > 0$)

$$\| D_\Delta \|_1 \geqq k \left(\frac{\log N}{\log \log N} \right)^{\frac{1}{4}}, \tag{8.2}$$

provided $N = | \Omega |$ (the cardinal number of Ω) is sufficiently large.

8.2 PROOF OF (8.2). Introduce H as the annihilator in G of Φ and identify in the usual way the dual of H with Γ / Φ. Likewise identify the dual of $K = G/H$ with Φ ([7], (24.11)).

We then have

$$\| D_\Delta \|_1 = \int_G \Big| \sum_{\gamma\in\Delta} \gamma \Big| \, d\lambda_G$$

$$= \int_{G/H} d\lambda_{G/H}(\bar{x}) \int_H \Big| \sum_{\theta\in\Omega} \sum_{\phi\in\Lambda} \theta\,(x+y)\,\phi\,(x+y) \Big| \, d\lambda_H(y),$$

the inner integral being viewed as a function of $\bar{x} = x+H$ Thus, writing $\bar{\theta}$ for $\pi\,(\theta)$ and noting that $\phi\,(y) = 1$ for $\phi \in \Lambda \subseteq \Phi$ and $y \in H$, we obtain

$$\| D_\Delta \|_1 = \int_{G/H} d\lambda_{G/H}(\bar{x}) \int_H \Big| \sum_{\theta\in\Omega} \alpha\,(\theta, \mathrm{x})\bar{\theta}\,(y) \Big| \, d\lambda_H(y), \tag{8.3}$$

where

$$\alpha\,(\theta, x) = \theta\,(x) \sum_{\phi\in\Lambda} \phi\,(x).$$

Now, since the dual of H (namely Γ/Φ) is torsion-free ([7], (A.4)), Theorem A of [8] shows that (for a suitable absolute constant $k > 0$) we have

$$\int_H \Big| \sum_{\theta\in\Omega} \alpha\,(\theta, x)\bar{\theta}\,(y) \Big| \, d\lambda_H(y) \geqq k\left(\frac{\log N}{\log\log N}\right)^{\frac{1}{4}} \min_{\theta\in\Omega} |\,\alpha\,(\theta, x)\,|$$

$$= k\left(\frac{\log N}{\log\log N}\right)^{\frac{1}{4}} \Big| \sum_{\phi\in\Lambda} \phi\,(\bar{x}) \Big|, \tag{8.4}$$

since $|\,\theta\,(x)\,| = 1$ and $\phi\,(x)$ depends only $\bar{x}$. By (8.3) and (8.4),

$$\| D_\Delta \|_1 \geqq k\left(\frac{\log N}{\log\log N}\right)^{\frac{1}{4}} \int_{G/H} \Big| \sum_{\phi\in\Lambda} \phi\,(\bar{x}) \Big| \, d\lambda_{G/H}(\bar{x}). \tag{8.5}$$

Since $\Lambda \neq \varnothing$, the remaining integral is not less than the maximum modulus of the Fourier transform of the function $\bar{x} \mapsto \sum_{\phi\in\Lambda} \phi\,(\bar{x})$, i.e., is not less than unity. Thus, (8.2) follows from (8.5).

8.3 PROOF OF 7.4 (i). The conclusions stated in case (i) of 7.4 are now almost immediate. If $\mathscr{D} = (\Delta_j)_{j\in N}$ is a grouping of infinite type covering Γ_0, $|\,\pi\,(\Delta_j)\,| \to \infty$ and so, since $\Lambda_j \subseteq \Phi$, $|\,\pi\,(\Omega_j)\,| \to \infty$. Then (8.2) shows that (7.6) is satisfied, and it remains only to refer to 7.6.

8.4 SUPPLEMENTARY REMARKS. The fact that, when G is not 0-dimensional, (7.6) holds for suitable subgroups Γ_0 of Γ and suitable groupings $\mathscr{D} = (\Delta_j)_{j\in N}$ covering Γ_0 can be derived without appeal to Theorem A of [8]. To do this, it suffices to take $\gamma_k \in \Gamma \setminus \Phi$ $(k = 1, 2, ..., m)$ such that the family $(\gamma_k)_{1 \leqq k \leqq m}$ is independent (see [7], (A.10)), define

$$\Gamma_0 = \{\textstyle\sum_{k=1}^m n_k \gamma_k : n_k \in Z \text{ for } k = 1, 2, ..., m\},$$

and make use of the formula

$$\int_G F(\gamma_1(x), ..., \gamma_m(x)) d\gamma_G(x)$$
$$= (2\pi)^{-m} \int_0^{2\pi} ... \int_0^{2\pi} F(e^{it}, ..., e^{it_m}) dt_1 ... dt_m, \tag{8.6}$$

valid for every $F \in C(T^m)$, where T denotes the circle group. (Recall that $\sum_{k=1}^m n_k \gamma_k$ denotes the character $x \mapsto \gamma_1(x)^{n_1} ... \gamma_m(x)^{n_m}$ of G.) It then appears that (7.6) holds when one takes

$$\Delta_j = \{\textstyle\sum_{k=1}^m n_k \gamma_k : |n_k| \leqq r_{j,k} \text{ for } k = 1, 2, ..., m\},$$

where the $r_{j,k}$ are positive integers satisfying $r_{j,k} < r_{j,k+1}$ and $\lim_{j\to\infty} r_{j,k} = \infty$. Moreover, when $m = 1$, the Cohen-Davenport result (essentially Theorem A of [8] for the case $G = T$) shows that (7.6) holds for every grouping $\mathscr{D}$ covering Γ_0.

The verification of (8.6) is simple. First note that, if G and G' are compact groups, and if ϕ is a continuous homomorphism of G into G', then

$$\int_G (F \circ \phi) d\lambda_G = \int F d\lambda_{\phi(G)} \tag{8.7}$$

for every $F \in C(G')$. (This is a consequence of the fact that $F \mapsto \int_G (F \circ \phi) d\lambda_G$ is invariant under translation by elements of $\phi(G)$, combined with the uniqueness of the normalised Haar measure on a compact group.) Taking $G' = T^m$ and $\phi : x \mapsto (\gamma_1(x), ..., \gamma_m(x))$, the stated conditions on the γ_k are just adequate to ensure that the annihilator in Z^m (identified in the canonical fashion with the dual of T^m) of $\phi(G)$ is $\{(0, ..., 0)\}$ and so ([7], (24.10)) that $\phi(G) = T^m$. Accordingly, (8.6) appears as a special case of (8.7).

It is perhaps worth indicating that special cases of (8.7) can be exploited in other ways. For example, suppose more generally that κ is an arbitrary nonvoid set and that $(\gamma_k)_{k\in\kappa}$ is a finite or infinite independent family of elements of $\Gamma \setminus \Phi$. Denote by Γ_0 the subgroup of Γ generated by $\{\gamma_k : k \in \kappa\}$. Taking $G' = T^\kappa$ and $\phi : x \mapsto (\gamma_k(x))_{k\in\kappa}$, one may use (8.7) in a similar fashion to show that there is an isometric isomorphism $F \leftrightarrow F \circ \phi = f$ between $L^p(T^\kappa)$ (or $C(T^\kappa)$) and the subspace of $L^p(G)$ (or $C(G)$) formed of those $f \in L^p(G)$ or $C(G)$) such that $\mathrm{sp}(f) \subseteq \Gamma_0$. Moreover, if one identifies in the canonical fashion the dual of T^κ with the weak direct product Z^{κ^*}, the said isomorphism is such that $\hat{F} = \hat{f} \circ \phi'$, where ϕ' is the isomorphism of Z^{κ^*} onto Γ_0 defined by $(n_k) \to \sum_{k \in \kappa} n_k \gamma_k$.

One consequence of this may be expressed roughly as follows: If the compact Abelian group G is such that $\Gamma \setminus \Phi$ contains an independent family of (finite or infinite) cardinality m, then Fourier series on G behave, in respect of convergence or summability, no better than do Fourier series on T^m.

Another consequence is that, if Δ is a subset of Γ_0, then Δ is a Sidon (or $\Lambda(p)$) subset of Γ if and only if $\phi'^{-1}(\Delta)$ is a Sidon (or $\Lambda(p)$) subset of Z^{κ^*}.

8.5 Further results. Theorem A of [8] implies something stronger than (8.2), namely: if ω is any complex-valued function on Γ such that

$$\omega(\gamma + \phi) = \omega(\gamma) \quad (\gamma \in \Gamma, \phi \in \Phi), \tag{8.8}$$

so that ω can be regarded as a function on Γ / Φ, and if we write

$$D_\Delta^\omega = \sum_{\gamma \in \Delta} \omega(\gamma)\, \bar{\gamma},\ S_\Delta^\omega f = \sum_{\gamma \in \Delta} \omega(\gamma)\, \hat{f}(\gamma), \tag{8.9}$$

then, for $\Delta = \Omega + \Lambda$ as in (8.1), we have

$$\| D_\Delta^\omega \|_1 \geqq k \left(\frac{\log N}{\log \log N} \right)^{\frac{1}{4}} \min_{\gamma \in \Omega} | \omega(\gamma) | \tag{8.10}$$

provided $N = | \Omega |$ is sufficiently large.

So, if we can arrange for $\Omega = \Omega_j$ to vary in such a way that the right-hand side of (8.10) tends to infinity with j, the substance of 7.6 will lead to a continuous f satisfying $\mathrm{sp}\,(f) \subseteq \Gamma_0$ and

$$\overline{\lim_{j \to \infty}}\ \mathrm{Re}\, S_{\Delta_j}^\omega f(0) = \infty. \tag{8.11}$$

Taking the most familiar case, in which $G = T$, $\Gamma = Z$ and $\Phi = \{0\}$, and supposing $\Delta = \Omega$ to range over a sequence (Δ_j) of finite subsets of Z such that, if $N_j = | \Delta_j |$,

$$\lim_j \left(\frac{\log N_j}{\log \log N_j} \right)^{\frac{1}{4}} \min_{n \in \Delta_j} | \omega(n) | = \infty,$$

the construction will lead to a continuous f on T such that

$$\overline{\lim_j}\ \mathrm{Re}\, S_{\Delta_j}^\omega f(0) = \infty.$$

In particular, taking $\Delta_j = \{n \in Z : 2^j \leqq n < 2^{j+1}\}$ it can be arranged that

$$\sum_{n \in Z} \frac{\pm \hat{f}(n)}{(\log(2+|n|))^{\alpha}}$$

diverges for any preassigned distribution of signs $\pm$ and any preassigned $\alpha < \frac{1}{4}$.

Of course, much stronger results are derivable by using random (and unspecifiable!) changes of sign, but there seems little hope of making this even remotely constructive.

§ 9. *Discussion of case (*ii*) :* G *0-dimensional*

9.1 In this case there is ([7], (7.7)) a base of neighbourhoods of zero in G formed of compact open subgroups W. For each such W the annihilator $\Delta = W^{\circ}$ in Γ of W is a finite subgroup of Γ. Define

$$k_W = \lambda_G(W)^{-1} \times \text{characteristic function of } W. \tag{9.1}$$

Then k_W is continuous, $k_W \geqq 0$, $\int_G k_W \, d\lambda_G = 1$. The transform $\hat{k}_W$ of k_W is plainly equal to unity on Δ. On the other hand, since W is a subgroup, we have for $a \in W$ and $\gamma \in \Gamma$

$$\begin{aligned} \hat{k}_W(\gamma) &= \int_G k_W(x) \overline{\gamma(x)} \, d\lambda_G(x) = \int_G k_W(x+a) \overline{\gamma(x)} \, d\lambda_G(x) \\ &= \int_G k_W(y) \overline{\gamma(y-a)} \, d\lambda_G(y) \\ &= \gamma(a) \hat{k}_W(\gamma), \end{aligned}$$

which shows that $\hat{k}_W(\gamma) = 0$ if $\gamma \in \Gamma \setminus \Delta$. Thus $\hat{k}_W$ is the characteristic function of Δ, and so

$$k_W = D_{W^{\circ}}. \tag{9.2}$$

By (9.1) and (9.2), a routine argument shows that, if $1 \leqq p < \infty$ and $f \in L^p(G)$, then

$$f = \lim_W S_{W^{\circ}} f \tag{9.3}$$

in $L^p(G)$; and that (9.3) holds uniformly for any continuous f.

9.2 Proof of 7.4 (ii). If Γ_0 is any countably infinite subgroup of Γ we can choose a sequence W_j of compact open subgroups of G such that

$W_{j+1} \subseteq W_j$ and $\Gamma_0 \subseteq \bigcup_{j=1}^{\infty} W_j^\circ$, where W_j° is a finite subgroup of Γ and $W_j^\circ \subseteq W_{j+1}^\circ$. The $\Delta_j = W_j^\circ \cap \Gamma_0$ satisfy (7.2) and, from (9.3),

$$f = \lim_j S_{\Delta_j} f \tag{9.4}$$

uniformly for any continuous f with $\mathrm{sp}(f) \subseteq \Gamma_0$. This verifies the statements made in 7.4 (ii).

9.3 By using the results in [3], more can be said in case (ii) of 7.4; cf. [3], Theorem (2.9) and Example (4.8).

Let $f \in L^1(G)$ and let Γ_0 be any countable subgroup of Γ containing $\mathrm{sp}(f)$. Choose the W_j as in 9.2. Then, apart from the fact that (W_j) is not in general a base at 0 in G (they can be chosen to be so if and only if G is first countable), (W_j) is an open-compact D''-sequence ([3], p. 188). The proof of Theorem (2.5) of [3] is easily modified to show that

$$f(x) = \lim_{j \to \infty} S_{W_j^\circ} f(x) \tag{9.5}$$

holds for almost all $x \in G$. Moreover, Theorem (2.7) of [3] applies to show that the majorant function

$$S^* f(x) = \sup_{j \in N} \left| S_{W_j^\circ} f(x) \right| \tag{9.6}$$

satisfies the estimates

$$\| S^* f \|_p \leqq 2 \left(p (p-1)^{-1}\right)^{\frac{1}{p}} \| f \|_p \quad (1<p<\infty) \tag{9.7}$$

$$\| S^* f \|_1 \leqq 2 + 2 \int_G |f| \log^+ |f| \, d\lambda_G, \tag{9.8}$$

$$\| S^* f \|_p \leqq 2 (1-p)^{\frac{1}{p}} \| f \|_1 \quad (0<p<1). \tag{9.9}$$

In particular, the convergence in (9.5) is dominated whenever

$$|f| \log^+ |f| \in L^1(G).$$

A more immediate consequence of (9.1) and (9.2) is a strong version of localisability of the convergence of Fourier series: if $f \in L^1(G)$ vanishes a.e. on some neighbourhood of $x_0 \in G$, we can choose the W_j so that $S_{\Delta_j} f(x_0) = 0$ for every sufficiently large j. [A suitable choice of W_j may be made once for all, independent of f, if G is first countable.] Nothing similar is true for general G; see, for example, [11], Vol. II, pp. 304-305.

§ 10. *Concerning the polynomials* Q_j.

There is no difficulty in making fairly explicit the construction of t.p.s Q_j of the type employed in 7.6.

For $p > 0$, $t \geqq 0$ define

$$h_p(t) = \begin{cases} 1 & \text{if } t \leqq p, \\ 2\left(1 - \dfrac{t}{2p}\right) & \text{if } p \leqq t \leqq 2p, \\ 0 & \text{if } t \geqq 2p. \end{cases} \tag{10.1}$$

For all complex z define

$$f_p(z) = \begin{cases} 0 & \text{if } z = 0, \\ |z|^{-1}\bar{z}\, h_p(|z|) & \text{if } z \neq 0. \end{cases} \tag{10.2}$$

Write

$$\left.\begin{aligned} E_n(z) &= \pi^{-1} n \exp(-n|z|^2, \\ P_{n,k}(z) &= \pi^{-1} n \sum_{j=0}^{k} \frac{(-1)^j}{j!} (n|z|^2)^j \end{aligned}\right\} \tag{10.3}$$

Let μ denote Lebesgue measure on C (identified with R^2 in the canonical fashion).

It is then routine to verify that

$$\left.\begin{aligned} &\| E_n * f_p \|_\infty \leqq \| f_p \|_\infty = 1, \\ &\lim_{n\to\infty} E_n * f_p = f_p \end{aligned}\right\} \tag{10.4}$$

uniformly on any compact set omitting 0. From this it follows that to every $p > 0$ and every positive integer ν correspond positive integers $\bar{n}(p, \nu)$, $\bar{k}(p, \nu)$ such that

$$\left.\begin{aligned} &\left| |z|^{-1}\bar{z} - f_p * P_{\bar{n},\bar{k}}(z) \right| \leqq \frac{1}{\nu} \text{ for } \frac{1}{\nu} \leqq |z| \leqq p, \\ &|f_p * P_{\bar{n},\bar{k}}(z)| \leqq 1 + \frac{1}{\nu} \text{ for } |z| \leqq p. \end{aligned}\right\} \tag{10.5}$$

Now

$$f_p * P_{\bar{n},\bar{k}}(z) = q_{p,\nu}(z, \bar{z}), \tag{10.6}$$

where

$$q_{p,\nu}(X, Y) = \pi^{-1}\,\bar{n}(p,\nu) \sum_{j=0}^{\bar{k}(p,\nu)} \frac{(-\bar{n}(p,\nu))^j}{j!} \sum_{l=0}^{j} \sum_{m=0}^{j} \binom{j}{l}\binom{j}{m} X^l Y^m$$

$$(-1)^{l+m} \int \zeta^{j-l}\bar{\zeta}^{j-m} f_p(\zeta)\, d\mu(\zeta)$$

$$= \sum_{l,m=0}^{\bar{k}(p,\nu)} C_{p,\nu}(l,m)\, X^l\, Y^m. \tag{10.7}$$

It is easily verifiable that the $C_{p,\nu}(l, m)$ are real-valued.

If θ is a bounded measurable function on G and

$$\mathring{Q}_{p,\nu} = q_{p,\nu}(\theta, \bar{\theta}),\ p \geqq \|\theta\|_\infty, \tag{10.8}$$

we have from (10.5)

$$\left.\begin{aligned} \Big|\, |\theta|^{-1}\bar{\theta} - \mathring{Q}_{p,\nu} \Big| &< \frac{1}{\nu} \text{ whenever } |\theta| \geqq \frac{1}{\nu}, \\ |\mathring{Q}_{p,\nu}| &\leqq 1 + \frac{1}{\nu} \text{ everywhere on } G. \end{aligned}\right\} \tag{10.9}$$

If θ is a t.p., then $\mathring{Q}_{p,\nu}$ is a t.p. and

$$\mathrm{sp}\,(\mathring{Q}_{p,\nu}) \subseteq [\mathrm{sp}\,(\theta)]. \tag{10.10}$$

From (10.9) we obtain

$$\Big|\, |\theta| - \theta\,\mathring{Q}_{p,\nu} \Big| \leqq \begin{cases} \nu^{-1}\,|\theta| \text{ whenever } |\theta| > \dfrac{1}{\nu}, \\ \left(2+\dfrac{1}{\nu}\right)|\theta| \text{ everywhere,} \end{cases}$$

whence it follows that, if $\theta \neq 0$,

$$\begin{aligned} \Big| \int_G \theta\,\mathring{Q}_{p,\nu}\, d\lambda G \Big| &\geqq (1-\nu^{-1})\,\|\theta\|_1 - \nu^{-1}(2+\nu^{-1}) \\ &\geqq (1-2\nu^{-\frac{1}{2}})\,\|\theta\|_1 \end{aligned} \tag{10.11}$$

provided $\nu \geqq 9\,\|\theta\|_1^{-2}$.

Taking $\theta = D_{\Delta_j}$ and $p_j \geqq \|D_{\Delta_j}\|$, the trigonometric polynomials

$$Q_j' = \left(1+\frac{1}{\nu}\right)^{-1} \mathring{Q}_{p_j,\nu} = \left(1+\frac{1}{\nu}\right)^{-1} q_{p_j,\nu}(D_{\Delta_j}, \bar{D}_{\Delta_j}) \tag{10.12}$$

are then seen from (10.9), (10.10) and (10.11) to satisfy

$$\left.\begin{aligned} \| Q_j' \| &\leqq 1, \\ \mathrm{sp}\,(Q_j') &\subseteq [\Delta_j], \\ \left| \int v\, D_{\Delta_j}\, Q_j'\, d\lambda_G \right| &\geqq (1 - 3v^{-\frac{1}{2}})\, \| D_{\Delta_j} \|_1 \end{aligned}\right\} \qquad (10.13)$$

provided v is chosen $\geqq 9 \| D_{\Delta_j} \|_1^{-1}$. In view of (7.6), we may choose the integer $v \geqq \max_j (36, 9 \| D_{\Delta_j} \|_1^{-1})$. Then (10.13) shows that there are unimodular complex numbers ξ_j such that the $Q_j = \xi_j Q_j'$ satisfy (7.7).

Appendix

Rudin-Shapiro sequences

A.1 Notations and definitions. As hitherto, all topological groups G are assumed to be Hausdorff; and, for any locally compact group G, λ_G will denote a selected left Haar measure, with respect to which the Lebesgue spaces $L^p(G)$ are to be formed. $C_c(G)$ denotes the set of complex-valued continuous functions on G having compact supports.

If X and Y are topological groups, Hom (X, Y) denotes the set of continuous homomorphisms of X into Y.

Suppose henceforth G to be locally compact. As in 5.1, if $k \in C_c(G)$, T_k will denote the convolution operator

$$f \mapsto f * k$$

with domain $C_c(G)$ and range in $C_c(G)$; and $\| k \|_{p,q}$ will denote the (p, q)-norm of this operator, i.e., the smallest real number $m \geqq 0$ such that

$$\| f * k \|_q \leqq m \| f \|_p \quad (f \in C_c(G)).$$

It is well-known that, if G is Abelian, $\| k \|_{2,2}$ is equal to

$$\| \hat{k} \|_\infty = \sup_{\gamma \in \Gamma} | \hat{k}(\gamma) |,$$

where Γ is the character group of G and $\hat{k}$ is the Fourier transform of k. (Something similar is true whenever G is compact, but we shall not use this.)

U-RS-sequences on G are as defined in 5.4.

In A.2-A.4 we are concerned with conditions on G sufficient to ensure the possibility of constructing U-RS-sequences on G for certain choices of U. In A.5 we use Rudin-Shapiro sequences on infinite compact Abelian groups to support statements made in 7.5.

A.2 THE ABELIAN CASE. If G is Abelian and nondiscrete, the methods of § 2 of [5] show how to construct (reasonably explicitly) a U-RS-sequence (h_n) on G for any preassigned nonvoid open $U \subseteq G$; see also [7], (37.19.b). In addition, we may assume that each $\hat{h}_n$ is integrable on Γ, the character group of G. [To see this, let V be a compact neighbourhood of the origin of G and let W be a nonvoid subset of U such that $V + W \subseteq U$. Let $\{u_i\}$ be an approximate identity on G comprised of functions in $C_c(G)$ with supports in V and Fourier transforms in $L^1(\Gamma)$. Finally, let (k_n) be a W-RS-sequence; then for each $n \in N$ we may select i_n so that $(k_n * u_{i_n})$ is a U-RS-sequence with the further property that $(k_n * u_{i_n})^\wedge = \hat{k}_n \hat{u}_{i_n} \in L^1(\Gamma)$, as required.] We take this construction for granted (but see A.5 below) and use it to show how to construct U-RS-sequences on certain non-Abelian groups G. The basis of the extension is a simple technique of passage from a quotient group to the original, the crucial step being A.3.2 below.

A.3 THE NOT-NECESSARILY ABELIAN CASE.

A.3.1 Assume here that K is a compact normal subgroup of G. Let λ_K be normalised so that $\lambda_K(K) = 1$; and let $\pi : x \mapsto \bar{x}$ denote the natural mapping of G onto G/K.

If $f \in C_c(G)$, the function f' on G/K defined by

$$f'(\bar{x}) = \int_K f(xt)\, d\lambda_K(t) \tag{A.1}$$

belongs to $C_c(G/K)$; cf. [7], (15.21). If $g \in C_c(G/K)$, $g \circ \pi \in C_c(G)$ and

$$(g \circ \pi)' = g. \tag{A.2}$$

If τ_a denotes left-translation by amount a, it is verifiable that $(\tau_a f)' = \tau_{\bar{a}} f'$. From this it follows that the disposable factors in λ_G and $\lambda_{G/K}$ can be mutually adjusted so that

$$\int_G f\, d\lambda_G = \int_{G/K} f'\, d\lambda_{G/K} \tag{A.3}$$

for $f \in C_c(G)$. Using (A.3), a direct calculation confirms that

$$(f * (k \circ \pi))' = f' * k \tag{A.4}$$

whenever $f \in C_c(G)$ and $k \in C_c(G/K)$.

Another consequence of (A.3) is that for $1 \leqq p \leqq \infty$

$$\|f\|_p \geqq \|f'\|_p \tag{A.5}$$

for every $f \in C_c(G)$; and that for $0 < p \leqq \infty$

$$\|f\|_p = \|f'\|_p \tag{A.6}$$

for every $f \in C_c(G;K)$, the set of $f \in C_c(G)$ which are constant on cosets modulo K.

A.3.2 Let $k \in C_c(G/K)$. Then

$$\| k \circ \pi \|_{p,q} \leqq \| k \|_{p,q}. \tag{A.7}$$

Proof. For $f \in C_c(G)$, $f * (k \circ \pi) \in C_c(G;K)$ and (A.6) gives

$$\|f * (k \circ \pi)\|_q = \|(f * (k \circ \pi))'\|_q,$$

which by (A.4)

$$\begin{aligned} &= \|f' * k\|_q \\ &\leqq \|f'\|_p \|k\|_{p,q} \\ &\leqq \|f\|_p \|k\|_{p,q}, \end{aligned}$$

the last step by (A.5). Whence (A.7).

A.3.3 If (h_n) is a V-RS-sequence on G/K and $U = \pi^{-1}(V)$, then $(h_n \circ \pi)$ is a U-RS-sequence on G.

Proof. In view of A.3.2 it suffices to note that

$$\begin{aligned} \operatorname{supp}(h_n \circ \pi) &= \pi^{-1}(\operatorname{supp} h_n) \\ &\subseteq \pi^{-1}(V), \\ \| h_n \circ \pi \|_\infty &= \| h_n \|_\infty, \\ \| h_n \circ \pi \|_2 &= \| h_n \|_2, \end{aligned}$$

the last two because of (A.6) and (A.2).

A.3.4 Suppose that K is a compact normal subgroup of G and that one can construct V-RS-sequences on G/K for any given nonvoid open $V \subseteq G/K$. Then one can construct U-RS-sequences on G for any given open subset U of G which contains K.

PROOF. Apply A.3.3, taking a nonvoid open subset W of G such that $KW \subseteq U$, and noting that $V = \pi(W)$ is then nonvoid and open in G/K and that $\pi^{-1}(V) = KW \subseteq U$.

A.3.5 Let $\delta(G)$ be the closure in G of the derived (= commutator) subgroup of G, and suppose that $\delta(G)$ is compact and nonopen in G. Then one can construct U-RS-sequences on G for any given open subset U of G containing $\delta(G)$. (Note that, since $\delta(G)$ is a closed subgroup of G, it is nonopen in G if and only if it has empty interior, or if and only if it is locally null for λ_G.)

PROOF. This follows from A.2 and A.3.4 because:
$\delta(G)$ is in any case a normal subgroup of G such that $G/\delta(G)$ is LCA [see [7], (5.22), (5.26), (23.8)]; and $\delta(G)$ is nonopen in G if and only if $G/\delta(G)$ is nondiscrete ([7], (5.21)).

A.3.6 The hypotheses of A.3.5 are satisfied in any one of the following cases (all groups being assumed Hausdorff and locally compact):

(i) $G = G_1 \times G_2$, where $\delta(G_1)$ and $\delta(G_2)$ are compact and $\delta(G_1)$ is nonopen in G_1 (hence in particular if $G = A \times B$, where A is nondiscrete Abelian and $\delta(B)$ is compact);

(ii) $\delta(G)$ is compact and there exists an open connected subset W of G such that $e \in W \nsubseteq \delta(G)$ (hence in particular if G is compact and connected and $\delta(G) \neq G$);

(iii) $\delta(G)$ is compact and, for some Abelian A, some $\varphi \in \mathrm{Hom}\,(G, A)$ and some connected open subset W of G, we have $e \in W$ and $\varphi \mid W$ nonconstant (hence in particular if G is compact and connected and $\mathrm{Hom}\,(G, A)$ is nontrivial);

(iv) $G = \varphi(H)$, where $\varphi \in \mathrm{Hom}\,(G, H)$ is such that $\mathrm{Ker}\,\varphi$ is locally countable (that is, such that $\mathrm{Ker}\,\varphi$ intersects each compact set in a countable set), and where $\delta(H)$ is compact and nonopen in H.

PROOF. (i) It is evident that $\delta(G) \subseteq \delta(G_1) \times \delta(G_2)$, which shows that $\delta(G)$ is compact and nonopen in G [if it were open, $\delta(G_1) = pr_{G_1}(\delta(G_1) \times \delta(G_2))$ would have interior points].

(ii) Were $\delta(G)$ to be open in G, W would be a disjoint union of $W \cap \delta(G)$ and $W \cap (G \setminus \delta(G))$, each relatively open in W. Since

$e \in W \cap \delta(G)$, connectedness of W would imply that $W \cap (G \not\setminus \delta(G)) = \varnothing$, i.e., $W \subseteq \delta(G)$, a contradiction.

(iii) Ker φ is a closed subgroup of G containing $\delta(G)$; since $W \nsubseteq$ Ker φ, it follows that $W \nsubseteq \delta(G)$. Now use (ii).

(iv) Clearly,

$$\delta(G) \subseteq \overline{\varphi\,(\delta(H))} = \varphi\,(\delta(H))$$

is compact. Suppose $\delta(G)$ were open in G. Then $\varphi(\delta(H))$ has interior points, and the same would be true of

$$\varphi^{-1}\,(\varphi\,(\delta(H))) = S\,\delta(H),$$

where $S =$ Ker φ. So there would exist a compact neighbourhood V of the identity in H such that

$$V \subseteq S\delta(H)$$

and so

$$V = V \cap (S\delta(H)).$$

But, if $y \in V \cap (S\delta(H))$, $y = sz$ for some $s \in S$ and $z \in \delta(H)$, hence $s = yz^{-1} \in V\delta(H)^{-1}$, and so $s \in (V\delta(H)^{-1}) \cap S$, which is countable by hypothesis, say $\{s_n : n \in N\}$. But then

$$y \in \bigcup_{n \in N} s_n\,\delta(H).$$

Thus

$$V = V \cap (S\delta(H)) \subseteq \bigcup_{n \in N} s_n\,\delta(H)$$

and so, since $\lambda_H(\delta(H)) = 0$,

$$0 < \lambda_H\,(V) \leqq \sum_{n \in N} \lambda_H\,(\delta(H)) = 0,$$

a contradiction.

A.3.7 Remarks. (i) A.3.6 (iii) suffices to show that any finite-dimensional unitary group $U(n)$ satisfies the hypotheses of A.3.5. [For $U(n)$ is compact and connected (see [7], (7.15)); and we may apply A.3.6 (iii) with $A = T$, the circle group, and $\varphi =$ det.]

On the other hand, it is easy to see (cf. A.3.6 (i) and its proof) that if $G = \prod_{i \in I} G_i$, where the G_i are compact and at least one of them satisfies the hypothesis of A.3.5, then G satisfies the said hypotheses.

So every product of unitary groups satisfies the hypotheses of A.3.5.

(ii) The hypotheses of A.3.5 are also satisfied if $G = G_1 \circledS G_2$, the semidirect product of G_1 and G_2 (see [7], (2.6) and (6.20)), provided G_1 is compact and $\delta(G_2)$ is compact and nonopen in G_2 (hence in particular if $G = A \circledS B$, where A is compact and B is nondiscrete and Abelian). In fact, $\delta(G) \subseteq G_1 \times \delta(G_2)$ and the proof proceeds as for A.3.6 (i).

A.4 THE OPERATORS $f \mapsto k * f$. Retaining the notations introduced in A.3, it turns out that (cf. (A.4))

$$((k \circ \pi) * f)' = k * f^{\vee\prime\vee} \tag{A.8}$$

for every $f \in C_c(G)$ and $k \in C_c(G/K)$, where, for any function g with domain a group X, $\check{g}$ denotes the function $x \mapsto g(x^{-1})$ with domain X. As a consequence, the results of A.3 have direct analogues for the operator $f \mapsto k * f$, provided G/K is unimodular, which is so if and only if G is unimodular.

A.5 CONCERNING 7.5.

A.5.1 Throughout A.5 we suppose G to be infinite compact Abelian. Let Γ_0 be any infinite subsemigroup of the character group Γ of G; $0 \in \Gamma_0$. The construction described in § 2 of [5] may be employed to produce t.p.s f_n ($n \in N$) on G which, together with their spectra S_n, satisfy the conditions:

$$\left.\begin{aligned} & S_0 = \{0\},\ S_n \subseteq \Gamma_0,\ |S_n| = 2^n \\ & B2^{n/2} \leqq \|f_n\|_s \leqq A2^{n/2} \quad (1 \leqq s \leqq \infty), \\ & \|f_n\|_{2,2} = \|\hat{f}_n\|_\infty \leqq 1, \\ & \hat{f}_n = \varphi \text{ on } S_n,\ 0 \text{ on } \Gamma \setminus S_n, \end{aligned}\right\} \tag{A.9}$$

where A and B are positive absolute constants and φ is a function on Γ with $\operatorname{Ran} \varphi \subseteq \{-1, 0, 1\}$ and $|\varphi(\gamma)| = 1$ if and only if $\gamma \in S_n$. (When $G = T$, these f_n are virtually the original Rudin-Shapiro t.p.s. In the terminology adopted in 5.4 above the $h_n = 2^{-n/2} f_n$ constitute a G-RS-sequence on G.)

If we now choose $\alpha_n \in \Gamma$ inductively so that, on writing $F_n = \alpha_n + S_n$, we have

$$\alpha_{n+1} \in \Gamma_0 \setminus [(F_0 \cup \ldots \cup F_n) - S_{n+1}],$$

then

$$\left.\begin{aligned} & |F_n| = |S_n| = 2^n,\ F_n \subseteq \Gamma_0, \\ & F_n \cap F_m = \varnothing \text{ if } m \neq n, \end{aligned}\right\} \tag{A.10}$$

and the t.p.s

$$w_n = 2^{-n/2} \alpha_n f_n \tag{A.11}$$

satisfy the relations

$$\left.\begin{aligned} \| w_n \|_\infty \leqq A,\ \hat{w}_n = 2^{-n/2} \varphi_n, \\ \operatorname{Ran} \varphi_n \subseteq \{-1, 0, 1\},\ | \varphi_n(\gamma) | = 1 \quad \text{if and only if} \quad \gamma \in F_n. \end{aligned}\right\} \tag{A.12}$$

From (A.10) and (A.12) it follows that at least one of the sets $A_n = \varphi_n^{-1}(\{1\})$, $B_n = \varphi_n^{-1}(\{-1\})$ has not fewer than 2^{n-1} elements. Define $\varepsilon_n = 1$, $C_n = A_n$ if $| A_n | \geqq 2^{n-1}$ and $\varepsilon_n = -1$, $C_n = B_n$ if $| A_n | < 2^{n-1}$. Then

$$\left.\begin{aligned} (\varepsilon_n w_n)^\wedge (\gamma) = 2^{-n/2} \quad \text{if} \quad \gamma \in C_n, \\ C_n \subseteq F_n,\ | C_n | \geqq 2^{n-1}. \end{aligned}\right\} \tag{A.13}$$

A.5.2 In terms of the construction given in A.5.1, it is possible to write down any number of continuous functions f on G and sequences (Δ_j) of finite subsets of Γ_0 such that

$$\left.\begin{aligned} \Delta_j \subseteq \Delta_{j+1}, \\ \operatorname{sp}(f) \subseteq \Gamma_0, \\ S_{\Delta_j} f(0) \quad \text{is real and} \quad \lim_{j \to \infty} S_{\Delta_j} f(0) = \infty, \\ \sum_{\gamma \in \Gamma} | \hat{f}(\gamma) | = \infty; \end{aligned}\right\} \tag{A.14}$$

cf. the statements made in 7.5.

Indeed, if $(c_n)_{n=0}^{\infty}$ is a sequence of real numbers satisfying

$$c_n \geqq 0, \sum_{n=0}^{\infty} c_n < \infty, \sum_{n=0}^{\infty} 2^{n/2} c_n = \infty, \tag{A.15}$$

and if

$$\Delta_j = C_0 \cup \ldots \cup C_j, \tag{A.16}$$

if suffices to take

$$f = \sum_{n=0}^{\infty} c_n \varepsilon_n w_n, \tag{A.17}$$

(A.14) being then a simple consequence of (A.12) and (A.13).

However, it is a consequence of the choice of the γ_n and α_n and of (A.12) [on evaluating the Fourier series of w_n at 0] that $\big| |A_n| - |B_n| \big| \leqq 2^{n/2}$, which implies that C_n contains only about one half the elements of F_n, so that $\bigcup_{j=1}^{\infty} \Delta_j$ falls far short of exhausting Γ_0. In particular, (Δ_j) is not a convergence grouping of the sort described in § 7.

A.5.3 Two further consequences of the construction in A.5.1 are perhaps worth mentioning in passing.

(i) For any complex-valued sequence $(c_n)_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} |c_n| < \infty, \tag{A.18}$$

the formula

$$g = \sum_{n=1}^{\infty} c_n w_n \tag{A.19}$$

yields a continuous function $g \in C(G)$. It is easy to specify choices of (c_n) in accord with (A.18), and of nonnegative functions η on Γ such that

$$\lim_{\gamma \to \infty} \eta(\gamma) = 0, \tag{A.20}$$

for which

$$\sum_{\gamma \in \Gamma} |\hat{g}(\gamma)|^{2 - 2\eta(\gamma)} = \infty. \tag{A.21}$$

One might, for example, take $c_n = n^{-2}$ and $\eta(\gamma) = 6n^{-1} \log n$ for $\gamma \in F_n$ $(n = 1, 2, \ldots)$ and $\eta(\gamma) = 0$ for $\gamma \in \Gamma \setminus F$, where $F = \bigcup_{n=1}^{\infty} F_n$.

This is an analogue of a well-known result of Banach for the case $G = T$; it provides numerous reasonably constructive counter-examples to conjectural improvements of the Hausdorff-Young theorem.

(ii) Take (c_n), η and g as in (i) immediately above. Let ψ be any nonnegative function on Γ which is bounded away from zero on F. Let further θ be any complex-valued function on Γ such that

$$\theta(\gamma) = \psi(\gamma) |\hat{g}(\gamma)|^{1 - 2\eta(\gamma)} . \operatorname{sgn} \hat{g}(\gamma) \quad \text{for} \quad \gamma \in F. \tag{A.22}$$

Then (A.21), (A.22) and Bochner's theorem combine to show that θ is not a Fourier-Stieltjes transform. Yet, if ψ is bounded, and if we define $\theta(\gamma) = 0$ for $\gamma \in \Gamma \setminus F$, (A.20) and the fact that $g \in C(G)$ ensure that

$$\theta \in \bigcap_{r > 2} l^r(\Gamma). \tag{A.23}$$

We thus obtain explicit examples of functions θ satisfying (A.23) which are not Fourier-Stieltjes transforms.

Note that, if every c_n is real and nonzero, an (unbounded) ψ can be chosen so as to make Ran $\theta = \{-1, 1\}$; this yields explicit examples of $\pm$ 1-valued functions θ which are not Fourier-Stieltjes transforms. (These are, of course, also obtainable by starting with functions sgn $\hat{h}$, where $h \in C(G)$, $\hat{h}$ is real-valued and $\hat{h} \notin l^1(\Gamma)$.)

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