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# TOURNAMENTS AND HADAMARD MATRICES

G. SZEKERES

*To the memory of J. Karamata*

1. A Hadamard matrix ( $H$ -matrix) is a square orthogonal matrix with all entries  $+1$  or  $-1$ . Apart from the trivial cases  $n = 1$  or  $2$ , the order of an  $H$ -matrix must be divisible by  $4$ , and it is a famous yet unsolved problem whether an  $H$ -matrix of order  $n = 4m$  exists for all  $m$ .

The construction of certain  $H$ -matrices can be achieved via tournaments. A tournament  $\mathcal{T}_n = \mathcal{T}(u_1, \dots, u_n)$  is a complete directed graph consisting of  $n$  nodes  $u_1, \dots, u_n$  and one directed edge  $u_i \xrightarrow{\quad} u_j$  for each pair of nodes. We write  $u_i \rightarrow u_j$  and say that  $u_i$  dominates  $u_j$ .  $N(\mathcal{T}_n)$  denotes the set of nodes of  $\mathcal{T}_n$ . For every subset  $\{v_1, \dots, v_k\}$  of  $N(\mathcal{T}_n)$  we define

$$S(v_1, \dots, v_k) = \{w \in N(\mathcal{T}_n); \quad w \rightarrow v_i, \quad i = 1, \dots, k\},$$

$$S'(v_1, \dots, v_k) = \{w' \in N(\mathcal{T}_n); \quad v_i \rightarrow w', \quad i = 1, \dots, k\}.$$

The dual  $\mathcal{T}'_n = \mathcal{T}(u'_1, \dots, u'_n)$  of  $\mathcal{T}_n$  is defined by the dominance rule

$$u'_i \rightarrow u'_j \Leftrightarrow u_j \rightarrow u_i$$

An automorphism of  $\mathcal{T}_n$  is a permutation  $\pi$  of its nodes which preserves orientation,  $u_i \rightarrow u_j \Leftrightarrow u_{\pi(i)} \rightarrow u_{\pi(j)}$ .

In an earlier paper [3] we have considered the following:

Property  $T_{k,m}$ : For every subset  $\{v_1, \dots, v_k\} \subset N(\mathcal{T}_n)$  of order  $k$ ,  $S(v_1, \dots, v_k)$  is at least of order  $m$ . A  $T_{k,m}$  tournament  $\mathcal{T}_n$  has order  $n \geq 2^k(m+1) - 1$  ([3], Lemma 3). We shall call it *extreme* if its order is exactly  $2^k(m+1) - 1$ . It is easily seen that for every  $m$  there exists an extreme  $T_{1,m}$  tournament (of order  $2m+1$ ). We shall examine here the existence of extreme  $T_{2,m}$  tournaments of order  $4m+3$  for special values of  $m$ . Interest in these tournaments stems from the fact that they supply  $H$ -matrices of order  $4m+4$ . In fact the sets  $S(u_i)$ ,  $i = 1, \dots, 4m+3$  have the property that each  $S(u_i)$  is of order  $2m+1$  and  $S(u_i) \cap S(u_j)$  for  $i \neq j$  is of order  $m$ , and from sets with this property one can immediately construct an  $H$ -matrix of order  $4m+4$  ([4], § 1). The converse is not neces-

arily true; there exist  $H$ -matrices and corresponding configurations of subsets with the above mentioned property which are not the sets  $S(u_i)$  of any tournament. I owe to Dr. N. Smythe the remark that the existence of extreme  $T_{2,m}$  tournaments is equivalent to the existence of “skew”  $H$ -matrices of order  $4m+4$ , that is  $H$ -matrices of the form  $I+S$  where  $I$  is the identity matrix and  $S$  is skew symmetric. I also owe to Dr. Smythe the proof of Lemma 3. The hitherto known orders of skew  $H$ -matrices are given by E. C. Johnsen in [5], Theorem 2.6. The present Theorem 6 gives infinitely many new orders; the first one is 76.

2. Lemma 1.

Let  $\mathcal{T}$  be a  $T_{2,m}$  tournament of order  $4m+3$ . Then

- (i)  $\mathcal{T}$  is regular, i.e.  $S(v), S'(v)$  are of order  $2m+1$  for every  $v \in N(\mathcal{T})$ .
- (ii)  $S(v_1, v_2)$  is of order  $m$  for every pair of nodes  $v_1, v_2 \in N(\mathcal{T})$ .
- (iii) The dual  $\mathcal{T}'$  of  $\mathcal{T}$  is also  $T_{2,m}$ .

These statements have been proved in [3] (Lemma 4).

Lemma 2.

Let  $\mathcal{T}(u_1, \dots, u_n)$  be  $T_{2,m}$  of order  $4m+3$ . Let  $u_i \rightarrow u_j$ ; then the set  $\{u_k; u_i \rightarrow u_k \rightarrow u_j\}$  is of order  $m$  and the set  $\{u_k; u_j \rightarrow u_k \rightarrow u_i\}$  is of order  $m+1$ .

*Proof.* The first set is identical with  $S'(u_i) - S'(u_i, u_j) - \{u_j\}$ , the second set is identical with  $S(u_i) - S(u_i, u_j)$ . The statement now follows from Lemma 1.

Theorem 1.

If there exists an extreme  $T_{2,m}$  tournament then there also exists an extreme  $T_{2,2m+1}$  tournament (of order  $8m+7$ ).

This is basically the well known duplication theorem of  $H$ -matrices though not an obvious consequence of it.

Let  $n = 4m+3$  and  $u_1, \dots, u_n$  the nodes of a  $T_{2,m}$  tournament  $\mathcal{T}_n$ . Write  $i \rightarrow j$  if  $u_i \rightarrow u_j$ . Let  $u'_1, \dots, u'_n$  be the nodes of a dual  $\mathcal{T}'_n$ . We define  $\mathcal{T} = \mathcal{T}_{2n+1}$  as containing the disjoint subtournaments  $\mathcal{T}_n, \mathcal{T}'_n$  and another node  $v$  with the following additional dominance rules:

$$(1) \quad v \rightarrow u'_i \rightarrow u_i \rightarrow v \quad \text{for} \quad i = 1, \dots, n.$$

Furthermore if  $i \rightarrow j$  then

$$(2) \quad u'_i \rightarrow u_j, \quad u_i \rightarrow u'_j.$$

These rules define  $\mathcal{T}$  completely; we show that  $\mathcal{T}$  is  $T_{2,2m+1}$ . We merely enumerate  $S(v_1, v_2)$  for all possible pairs of nodes of  $\mathcal{T}$ .

$$S(v, u_i) = \{u_k; k \rightarrow i\},$$

$$S(v, u'_i) = \{u_k; k \rightarrow i\} \text{ are of order } 2m+1 \text{ by Lemma 1 (i).}$$

$$S(u_i, u_j) = \{u_k; k \rightarrow i, k \rightarrow j\} \text{ order } m \text{ by Lemma 1 (ii)}$$

$$\cup \{u'_k; k \rightarrow i, k \rightarrow j\} \text{ order } m$$

$$\cup \{u'_i\} \text{ if } i \rightarrow j$$

$$\{u'_j\} \text{ if } j \rightarrow i \text{ order } 1.$$

$$S(u'_i, u'_j) = \{u'_k; i \rightarrow k, j \rightarrow k\} \text{ order } m \text{ by Lemma 1 (iii)}$$

$$\cup \{u_k; k \rightarrow i, k \rightarrow j\} \text{ order } m$$

$$\cup \{v\} \text{ order } 1.$$

$$S(u_i, u'_i) = \{u_k; k \rightarrow i\} \text{ order } 2m+1$$

$$S(u_i, u'_j) = \{u_k, k \rightarrow i, k \rightarrow j\} \text{ order } m$$

$$\cup \{u'_k; j \rightarrow k, k \rightarrow i\} \text{ order } m+1 \text{ if } i \rightarrow j$$

$$\text{order } m \text{ if } j \rightarrow i, \text{ by Lemma 2}$$

$$\cup \{u'_i\} \text{ if } j \rightarrow i \text{ order } 1.$$

The proof of Theorem 1 suggests that we should seek the existence of  $T_{2,m}$  tournaments  $\mathcal{T}_n$ ,  $n = 4m+3$ , with the following structure:

(E1)  $\mathcal{T}_n$  contains two disjoint dual subtournaments  $\mathcal{T}_{2m+1} = \mathcal{T}(u_\alpha; \alpha \in G)$ ,  $\mathcal{T}'_{2m+1} = \mathcal{T}(u'_\alpha; \alpha \in G)$ , indexed by an additive abelian group  $G$  of order  $2m+1$ , and another node  $v$ , such that

$$(E2) \quad u_\alpha \rightarrow v \rightarrow u'_\alpha, \quad \text{all } \alpha \in G,$$

$$(E3) \quad u_\alpha \rightarrow u_\beta \Rightarrow u_{\alpha+\gamma} \rightarrow u_{\beta+\gamma},$$

$$u_\alpha \rightarrow u'_\beta \Rightarrow u_{\alpha+\gamma} \rightarrow u'_{\beta+\gamma}, \quad \text{all } \gamma \in G.$$

Thus the regular representation of  $G$  acts as a group of automorphisms of  $\mathcal{T}_{2m+1}$ . We shall refer to conditions (E1)–(E3) as property (E).

A tournament  $\mathcal{T}_{4m+3}$  with property (E) is completely described by two sets of elements of  $G$ , namely

$$A = \{ \alpha; \alpha \neq 0, u_\alpha \rightarrow u_0 \},$$

$$B = \{ \beta; u_\beta \rightarrow u'_0 \}.$$

From (E3) it then follows that

$$(E3.1) \quad u_{\gamma+\alpha} \rightarrow u_\gamma$$

$$(E3.2) \quad u'_{\gamma-\alpha} \rightarrow u'_\gamma$$

$$(E3.3) \quad u_{\gamma+\beta} \rightarrow u'_\gamma$$

$$(E3.4) \quad u'_{\gamma-\beta'} \rightarrow u_\gamma$$

all

$$\gamma \in G, \quad \alpha \in A, \quad \beta \in B, \quad \beta' \in B' = G - B.$$

In order that (E3.1) be consistent, i.e. that  $u_{\gamma+\alpha} \rightarrow u_\gamma$  and  $u_\gamma \rightarrow u_{\gamma+\alpha}$  be mutually exclusive, it is necessary and sufficient that

$$(E4) \quad \alpha \in A \Leftrightarrow -\alpha \notin A.$$

In particular  $A$  must contain exactly  $m$  elements.

We wish to set up conditions for  $\mathcal{T}_{4m+3}$  to be  $T_{2,m}$ . We must examine  $S(v_1, v_2)$  for all possible pairs of nodes of  $\mathcal{T}_{4m+3}$ .

$S(v, u_\gamma) = \{ u_{\gamma+\alpha}; \alpha \in A \}$  by (E1) and (E3.1) hence is of order  $m$ , as required.

$S(v, u'_\gamma) = \{ u_{\gamma+\beta}; \beta \in B \}$  by (E1) and (E3.3), thus  $B$  must also contain exactly  $m$  elements (hence  $B' = G - B$  contains  $m+1$  elements).

$$S(u_{\gamma_1}, u_{\gamma_2}) = \{ u_{\gamma_1+\alpha_1} = u_{\gamma_2+\alpha_2}; \alpha_1, \alpha_2 \in A \} \\ \cup \{ u'_{\gamma_1-\beta_1} = u'_{\gamma_2-\beta_2}; \beta_1, \beta_2 \in B' \}.$$

Thus for  $\mathcal{T}_{4m+3}$  to be  $T_{2,m}$  it is necessary that for each  $\delta = \gamma_1 - \gamma_2 \neq 0$ , the total number of solutions of

$$(3.1) \quad \delta = \alpha_2 - \alpha_1, \quad \alpha_1, \alpha_2 \in A,$$

$$(3.2) \quad \delta = \beta'_1 - \beta'_2, \quad \beta'_1, \beta'_2 \in B',$$

be  $m$ . We show that this condition is also sufficient.

### Theorem 2.

In order that two subsets  $A = \{ \alpha_1, \dots, \alpha_m \}$ ,  $B = \{ \beta_1, \dots, \beta_m \}$  of  $G$ , both of order  $m$ , define a  $T_{2,m}$  tournament  $\mathcal{T}_{4m+3}$  with property (E), it is necessary and sufficient that

- (i)  $\alpha \in A \Leftrightarrow -\alpha \notin A$ , and
- (ii) for each  $\delta \in G$ ,  $\delta \neq 0$  the equations (3.1) and (3.2) should have altogether  $m$  distinct solutions.

We have already seen that the conditions are necessary. To prove sufficiency we have to show that the sets  $S(u'_{\gamma_1}, u'_{\gamma_2})$ ,  $S(u_{\gamma_1}, u_{\gamma_2})$  contain  $m$  elements. Now

$$\begin{aligned}
 S(u'_{\gamma_1}, u'_{\gamma_2}) &= \{u_{\gamma_1+\beta_1} = u_{\gamma_2+\beta_2}; \beta_1, \beta_2 \in B\} \\
 &\cup \{u'_{\gamma_1-\alpha_1} = u'_{\gamma_2-\alpha_2}; \alpha_1, \alpha_2 \in A\} \\
 &\cup \{v\}, \\
 S(u_{\gamma_1}, u_{\gamma_2}) &= \{u_{\gamma_1+\alpha} = u_{\gamma_2+\beta}; \alpha \in A, \beta \in B\} \\
 &\cup \{u'_{\gamma_1-\beta'} = u'_{\gamma_2-\alpha}; \alpha \in A, \beta' \in B'\}.
 \end{aligned}$$

But for  $\delta = \gamma_1 - \gamma_2 \in G$  the total number of solutions of  $\delta = \beta - \alpha$ ,  $\delta = \beta' - \alpha$ ,  $\alpha \in A$ ,  $\beta \in B$ ,  $\beta' \in B'$  is equal to the number of elements in  $A$  since  $\delta + \alpha$  is either  $\beta$  or  $\beta'$ . Hence  $S(u_{\gamma_1}, u_{\gamma_2})$  contains  $m$  elements. On the other hand for  $\delta = \gamma_1 - \gamma_2 \neq 0$  the total number of solutions of  $\delta = \beta_2 - \beta_1$ ,  $\delta = \alpha_1 - \alpha_2$  is  $m-1$ , by (3.1) and (3.2) and by the following Lemma (with  $k = m$ ,  $n = 2m+1$ ):

*Lemma 3.*

Let  $B = \{\beta_1, \dots, \beta_k\}$ ,  $B' = \{\beta'_1, \dots, \beta'_{n-k}\}$  be a partition of an abelian group  $G$  of order  $n$  into two disjoint subsets. For fixed  $\gamma \in G$  denote by  $N(\gamma)$ ,  $N'(\gamma)$  the number of solutions of the equations

$$\gamma = \beta_i - \beta_j, \quad \gamma = \beta'_i - \beta'_j,$$

respectively. Then

$$N'(\gamma) - N(\gamma) = n - 2k.$$

*Proof.* Form the sums  $\gamma + \beta_j$ ,  $j = 1, \dots, k$ . If  $r$  of these sums are in the set  $B$  then  $k-r$  are in the set  $B'$ ; consequently the number of sums  $\gamma + \beta'_j$ , in  $B'$  is  $(n-k) - (k-r) = n - 2k + r$ . But then  $N(\gamma) = r$ ,  $N'(\gamma) = n - 2k + r$ .

Two subsets  $A$  and  $B$  of an additive abelian group  $G$  of order  $2m+1$  will be called *complementary difference sets* in  $G$  if

(D0)  $A$  contains  $m$  elements,

(D1)  $\alpha \in A \Rightarrow -\alpha \notin A$ , and

(D2) for each  $\delta \in G$ ,  $\delta \neq 0$  the equations

$$\delta = \alpha_1 - \alpha_2, \quad \delta = \beta_1 - \beta_2$$

have altogether  $m-1$  distinct solution vectors

$$(\alpha_1, \alpha_2) \in A \times A, \quad (\beta_1, \beta_2) \in B \times B.$$

From conditions (D0) and (D1) it follows that  $0 \notin A$ . From condition (D2) it follows that also  $B$  must contain  $m$  elements. Furthermore by Lemma 3, (D2) is equivalent to the condition that (3.1) and (3.2) have altogether  $m$  distinct solution vectors  $(\alpha_1, \alpha_2) \in A \times A$ ,  $(\beta'_1, \beta'_2) \in B' \times B'$  where  $B' = G - B$ . Our main purpose is to demonstrate the existence of complementary difference sets when (i)  $4m+3$  is a prime power, (ii)  $2m+1$  is a prime power  $\not\equiv 1 \pmod{8}$ . In the case when  $2m+1$  is a prime power  $\equiv 1 \pmod{8}$ , a general existence theorem does not seem to hold; a machine search by David Blatt at Sydney University has shown that in the lowest non-trivial case  $m = 8$  there do not exist any complementary difference sets in the cyclic group of order 17.

3. We now pass to the construction of complementary difference sets in the cases indicated.

*Theorem 3.*

*If  $q = 4m+3$  is a prime power and  $G$  the cyclic group of order  $2m+1$  then there exist complementary difference sets in  $G$ .*

*Corollary. If  $q = 4m+3$  is a prime power then there exists a  $T_{2,m}$  tournament of type (E) and order  $q$ .*

*Proof.* Let  $\rho$  be a primitive root of  $GF(q)$ ,  $Q = \{ \rho^{2\beta}; \beta = 1, \dots, 2m+1 \}$  the set of quadratic residues in  $GF(q)$ . Define  $A$  and  $B$  by the rules

$$(4.1) \quad \alpha \in A \quad \text{iff} \quad \rho^{2\alpha} - 1 \in Q,$$

$$(4.2) \quad \beta \in B \quad \text{iff} \quad \rho^{2\beta} - 1 \in Q.$$

Since

$$-1 = \rho^{2m+1} \notin Q,$$

$$\rho^{2\alpha} - 1 \in Q \Leftrightarrow \rho^{-2\alpha} - 1 = -\rho^{-2\alpha}(\rho^{2\alpha} - 1) \notin Q$$

so that  $\alpha \in A \Rightarrow -\alpha \notin A$ , and conditions (D0) and (D1) are satisfied. Also

$$(4.3) \quad \beta' \in B' \quad \text{if} \quad -(\rho^{2\beta'} + 1) \in Q.$$

Suppose now that

$$(5.1) \quad \delta = \alpha_2 - \alpha_1 \neq 0, \quad \alpha_1, \alpha_2 \in A$$

where

$$(5.2) \quad \rho^{2\alpha_1} = 1 + \rho^{2(\lambda_1 - \delta)},$$

$$(5.3) \quad \rho^{2\alpha_2} = 1 + \rho^{2\lambda_2}$$

by (4.1) for suitable  $\lambda_1, \lambda_2 \in G$ . Then

$$\rho^{2\alpha_2} = \rho^{2(\alpha_1 + \delta)} = \rho^{2\delta} + \rho^{2\lambda_1}$$

by (5.1) and (5.2), hence by (5.3)

$$(5.4) \quad \rho^{2\delta} - 1 = \rho^{2\lambda_2} - \rho^{2\lambda_1}$$

where  $\rho^{2\lambda_2} + 1 \in Q$  by (5.3).

Similarly if

$$(5.1') \quad \delta = \beta'_2 - \beta'_1 \neq 0, \quad \beta'_1, \beta'_2 \in B'$$

where

$$(5.2') \quad -\rho^{2\beta'_1} = 1 + \rho^{2(\lambda_1 - \delta)}$$

$$(5.3') \quad -\rho^{2\beta'_2} = 1 + \rho^{2\lambda_2}$$

for some  $\lambda_1, \lambda_2 \in G$ , we get

$$-\rho^{2\beta'_2} = -\rho^{2(\delta + \beta'_1)} = \rho^{2\delta} + \rho^{2\lambda_1}$$

hence again

$$\rho^{2\delta} - 1 = \rho^{2\lambda_2} - \rho^{2\lambda_1}$$

with  $-(\rho^{2\lambda_2} + 1) \in Q$  by (5.3').

Conversely to every solution  $\lambda_1, \lambda_2 \in G$  of equation (5.4) we can determine uniquely  $\alpha_2 \in A$  or  $\beta'_2 \in B'$  from (5.3) or (5.3') depending on whether  $1 + \rho^{2\lambda_2} = \rho^{2\delta} + \rho^{2\lambda_1}$  is in  $Q$  or not, hence  $\alpha_1$  or  $\beta'_1$  from (5.1), (5.1') so that

also (5.2) or (5.2') be satisfied, implying  $\alpha_1 \in A$ ,  $\beta'_1 \in B'$ . Thus the total number of solutions of (5.1) and (5.1') is equal to the number of solutions of (5.4) which is  $m$  by the following Lemma (with  $\gamma = \rho^{2\delta} - 1$ ):

*Lemma 4.*

*Given  $\gamma \in GF(q)$ ,  $\gamma \neq 0$ ,  $q = 4m+3$ , the equation*

$$(6) \quad \gamma = \sigma_2 - \sigma_1$$

*has exactly  $m$  distinct solution vectors  $(\sigma_1, \sigma_2) \in Q \times Q$ .*

This is a well known result on perfect difference sets, e.g. Ryser [2], p. 133 in the case of  $q$  prime. We give here a brief proof, to prepare the ground for Theorem 5 where a similar but more involved argument will be used.

Denote by  $N(\gamma)$  the number of solutions  $(\sigma_1, \sigma_2) \in Q \times Q$  of (6) and consider the equations

$$(6.1) \quad 1 = \sigma_2 - \sigma_1$$

$$(6.2) \quad -1 = \sigma'_2 - \sigma'_1,$$

$\sigma_1, \sigma_2, \sigma'_1, \sigma'_2 \in Q$ . Each solution of (6.1) yields, by multiplication with  $\gamma_o \in Q$ , a solution of (6) with  $\gamma = \gamma_o$ , and conversely each solution of (6) with  $\gamma = \gamma_o \in Q$  yields, by multiplication with  $\gamma_o^{-1}$ , a solution of (6.1). Hence  $N(\gamma_o) = N(1)$  for each  $\gamma_o \in Q$ , and similarly  $N(-\gamma_o) = N(-1)$ . On the other hand  $1 = \sigma_2 - \sigma_1 \Leftrightarrow -1 = \sigma'_2 - \sigma'_1$  with  $\sigma'_2 = \sigma'_1 = \sigma_2$  hence also  $N(1) = N(-1)$  and we conclude (since each  $\gamma \neq 0$  is either  $\gamma_o$  or  $-\gamma_o$ ) that  $N(\gamma)$  is the same number  $\mu$  for each  $\gamma \neq 0$ . Therefore  $\mu(q-1) = 2\mu(2m+1)$  is equal to the number of expressions  $\sigma_1 - \sigma_2 \neq 0$ ,  $\sigma_1, \sigma_2 \in Q$  i.e. to  $2m(2m+1)$ , giving  $\mu = m$ .

*Theorem 4.*

*Let  $q = 4m+3$  be a prime power  $p^k$  and  $G$  the elementary abelian  $p$ -group of order  $p^k$  and exponent  $p$ . Then there exist complementary difference sets in  $G$ .*

*Corollary. If  $q = 4m+3$  is a prime power then there exists a  $T_{2,2m+1}$  tournament of type (E) and order  $2q+1$ .*

The proof follows immediately from Paley's construction of  $H$ -matrices of order  $q$  and the doubling described in Theorem 1. The group  $G$  of Theorem 4 is isomorphic to the additive group of  $GF(q)$  and we can use the elements of  $GF(q)$  to represent  $G$ . As before we denote by  $Q$  the set of quadratic residues of  $GF(q)$  and set  $A = B = Q$ ; then (D1) is trivially

satisfied and also (D2) (with  $m$  being replaced by  $2m+1$ ) since by Lemma 4 both equations  $\delta = \alpha_1 - \alpha_2$  ( $\alpha_1, \alpha_2 \in A = Q$ ) and  $\delta = \beta_1 - \beta_2$  ( $\beta_1, \beta_2 \in B = Q$ ) have  $m$  solutions.

*Theorem 5.*

Let  $q = 2m+1$  be a prime power  $p^k \equiv 5 \pmod{8}$  (hence  $m \equiv 2 \pmod{4}$ ) and  $G$  the elementary abelian  $p$ -group of order  $p^k$  and exponent  $p$ . Then there exist complementary difference sets in  $G$ .

*Corollary.* If  $q = 2m+1$  is a prime power  $\equiv 5 \pmod{8}$  then there exists a  $T_{2,m}$  tournament of order  $4m+3 = 2q+1$  and type (E).

An immediate consequence is

*Theorem 6.*

For  $q$  prime power  $\equiv 5 \pmod{8}$  there exists a skew Hadamard matrix of order  $2(q+1)$ .

Although Hadamard matrices of order  $2(q+1)$  are known to exist even when  $q \equiv 1 \pmod{8}$  (Paley [1], Lemma 4) the result in Theorem 6 seems to be new. Paley's matrices are not skew and it is very unlikely that their rows and columns can be rearranged so as to yield skew  $H$ -matrices and  $T_{2,m}$  tournaments. The configurations obtained from the present construction are definitely not isomorphic to those of Paley, except when  $q = 5$ .

*Proof of Theorem 5.* We again identify  $G$  with the additive group of  $GF(q)$ . Let  $\rho$  be a primitive root of  $GF(q)$  and  $G_0$  the multiplicative group of  $GF(q)$ , of order  $q-1$  and generated by  $\rho$ . Denote by  $H_0 = \langle \rho^4 \rangle$  the subgroup of index 4 of  $G_0$ ,  $H_i, i = 1, 2, 3$  the coset mod  $H_0$  in  $G_0$  containing  $\rho^i$ , and set  $K = H_0 \cup H_1, K^* = H_0 \cup H_3$ .

We take  $A = K, B = K^*$ . Both contain  $m$  elements since  $H_0$  contains  $\frac{1}{4}(q-1) = \frac{1}{2}m$  elements. Also condition (D1) is satisfied since  $-1 = \rho^{\frac{1}{2}(q-1)} = \rho^m \in H_2$  by assumption hence  $\alpha \in K \Rightarrow -\alpha \in H_2 \cup H_3$ .

To verify condition (D2) consider for fixed  $\delta_0 \in H_0$  the following equations in  $\alpha_1, \alpha_2 \in K, \beta_1, \beta_2 \in K^*$ :

$$(7.0) \quad \delta_0 = \alpha_1 - \alpha_2$$

$$(7.1) \quad \rho\delta_0 = \beta_1 - \beta_2$$

$$(7.2) \quad \rho^2\delta_0 = \alpha_1 - \alpha_2$$

$$(7.3) \quad \rho^3\delta_0 = \beta_1 - \beta_2.$$

Clearly the number of solutions of each of these equations is independent

of the choice of  $\delta_0 \in H_0$  since

$$\alpha \in K, \quad \beta \in K^* \Rightarrow \rho^{4i} \alpha \in K, \quad \rho^{4i} \beta \in K^*$$

for every  $i$ . Furthermore the numbers of solutions of (7.0) and (7.3) are equal to each other because  $\alpha \in K \Rightarrow \beta = \alpha \rho^3 \in K^*$  and  $\beta \in K^* \Rightarrow \rho^{-3} \beta = \alpha \in K$ . Similarly the numbers of solutions of (7.1) and (7.2) are equal because

$$\beta \in K^* \Rightarrow \rho \beta^* \in K.$$

Finally (7.0) and (7.2) have the same number of solutions because

$$\alpha \in K \Rightarrow -\rho^2 \alpha \in K.$$

By the same argument it can be shown that the number of solutions of each of the equations

$$(8.0) \quad \delta_0 = \beta_1 - \beta_2$$

$$(8.1) \quad \rho \delta_0 = \alpha_1 - \alpha_2$$

$$(8.2) \quad \rho^2 \delta_0 = \beta_1 - \beta_2$$

$$(8.3) \quad \rho^3 \delta_0 = \alpha_1 - \alpha_2$$

is the same. Hence for each  $\delta \neq 0$  the total number of solutions of

$$\delta = \alpha_1 - \alpha_2, \quad \delta = \beta_1 - \beta_2$$

is the same number  $\mu$ . Therefore  $\mu(q-1) = 2\mu m$  is equal to the total number of expressions  $\alpha_1 - \alpha_2, \beta_1 - \beta_2$ , i.e. to  $2m(m-1)$ , giving  $\mu = m-1$  as required.

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