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THE REAL NUMBER SYSTEM REVIEWED

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To the memory of J. Karamata

In the teaching of elementary mathematics there are several major problems that continue to demand attention. One of the most important and difficult of these is the problem of introducing the real numbers and establishing their characteristic properties. Indeed, the difficulty of this problem has led many mathematicians to believe that a mathematically adequate treatment of the real numbers should be put off until the university stage is reached. This decision requires that at the secondary level the student learn some hard and relatively sophisticated techniques (e.g. of calculating with decimals) and that he accept a great deal more on faith, as a matter of intuition (e.g. the theory of limits). Some writers on elementary mathematics even find themselves trapped in a vicious circle: they send their students to geometry for an intuitive grasp of the real number concept, but they then base their axiomatic treatment of geometry on a prior knowledge of the real number system! While circular arguments may be useful, even indispensable, in heuristic discussions, they also create a clear and immediate danger of confusing the student — and, still worse, the teacher himself. Instead of evading or postponing the issue, it is therefore highly desirable that a solution to this problem be sought in the context of secondary school mathematics.

Now " solving " a problem of mathematical pedagogy involves at least three considerations of major importance: the motivation for the mathematical concepts to be taught; the selection of a mathematically clear, simple and efficient way of developing those concepts; and the elaboration of a detailed teaching program through which the student's conceptual understanding and technical mastery may be built up step by step. Educators have very often slighted the first two considerations in order to concentrate on the design of the teaching program. In other words, they have accepted without reexamination an approach currently popular in mathematical circles, and have based their teaching programs upon it. There is good reason to think that the curricular reforms being proposed and tried out all over the world today would benefit by more mathematical study of the subject matter, aimed at finding new and simpler ways of treating it.

The case of the real number system seems to me to give particular point to these observations. At first sight, it might appear that there is little choice as to the way in which the introduction of the real numbers is to be motivated and carried out. The original and most compelling reason for inventing and using the real numbers was the need for a systematic process of measurement; and the resulting theory inevitably includes the definitions and techniques on which the study of the real numbers rests. Indeed, the ideas of Eudoxus on geometrical measurement, as expounded in the fifth book of Euclid's " Elements ", contain in germ all the essentials of a theory of real numbers [1]. What was lacking in the treatment of Eudoxus was supplied centuries later by Dedekind, who introduced the relevant concepts of set theory and used them to define and investigate the real numbers as cuts in the system of rational numbers [2]. Variants of this approach, such as the use of Cauchy sequences in place of cuts, have since been developed [3]. However, all these modern theories are very difficult to introduce into secondary school mathematics, chiefly because the definitions given for the real numbers are too involved and lead at once to rather intricate manipulations. Thus it appears to be worthwhile to take a new look at the ideas of Eudoxus and Dedekind in hopes of uncovering a simpler way of developing the real number system through an approach which will be more direct and less encumbered with complicated techniques.

Now as a matter of fact it turns out that the essential ingredients of such a new theory can be found in some fairly recent studies, beginning with a paper of A. N. Kolmogorov in 1946 [4] and a later but quite independent contribution of F. A. Behrend in 1956 [5]. The clarity and scope of the theory benefit in important ways from a current reexamination of the theory of measurement due to Krantz, Luce, Suppes, and Tversky [6]. In particular, a theorem of Krantz [7] brings some of Behrend's ideas to their full simplicity and effectiveness and strengthens the bonds between the theory of measurement and the theory of real numbers. It seems to me that this treatment of the real numbers, apart from being of intrinsic mathematical interest, could be exploited pedagogically to great advantage.

To describe the theory in brief, it is convenient to start with an archimedean-ordered abelian semi-group G. In the manner of Eudoxus we define an injection Φ of the strictly positive elements of G into the set of sequences of natural numbers by putting $\Phi: x \to \xi_x = \xi_{x,u}$, where $\xi_{x,u}(n) = \max \{m; mu < nx\}$ with $u > 0$ and $n \geq 1$. It is easy to see that the sequence ξ enjoys the following properties:

(1) $k\xi(n) \leq \xi(kn) < k(\xi(n) + 1)$

(2) for each n there exists k such that $k\xi(n) < \xi(kn)$.

With Kolmogorov we now define a sequence ξ to be a positive real number if and only if it has properties (1) and (2); and we denote the set of such sequences as R_+. Among the positive real numbers we may distinguish those which arise when we start with $G = N$, where N is the additive semi-group of the natural numbers. We call these numbers the positive rational numbers, and denote as Q_+ the set constituted by them. When we consider the natural partial ordering $<$ and the natural addition $+$ for sequences of natural numbers, we see at once that R_+ is not closed under this addition but does turn out to be a completely, linearly ordered set. Consequently we may define addition for the positive real numbers by putting $\xi \oplus \eta = \inf \{\zeta; \xi + \eta \leq \zeta\}$. It is now easy to verify that Φ is an order-preserving isomorphism of G into the ordered additive system R_+, a result which goes back to O. Hölder in 1904 [8]. When $G = N$ we see that $\xi_{p,u} \oplus \xi_{q,u} = \xi_{p+q,u}$, that $\xi_{kp,ku} = \xi_{p,u}$, and that $\xi_{p,u} < \xi_{q,u}$ if and only if $p < q$. The properties of the rational number system Q_+ are thereby established, so that we are led to write $\frac{p}{u}$ for $\xi_{p,u}$ in the sequel. An easy analysis shows that $\frac{p}{u} < \xi$ if and only if $pn < \xi(n)u$ for some n. In consequence, we have

$$\frac{\xi(n)}{n} < \xi \leqq \frac{\xi(n) + 1}{n}$$

and can quickly develop the technique of approximating real numbers by rational numbers, reflecting the fact that Q_+ is ordinally dense in R_+. Some important features of the algebraic structure of R_+ emerge at once. They can be summed up by asserting that R_+ is a completely ordered abelian semi-group, viz. $\oplus$ is a commutative and associative operation, and the equation $\xi \oplus \alpha = \beta$ has a (unique) solution in R_+ if and only if $\alpha < \beta$. Furthermore, the injection Φ is uniquely determined as the order-preserving isomorphism which maps u into $\frac{1}{1}$.

The further study of R_+ can be made to depend on the study of its order-preserving endomorphisms, as was pointed out by Behrend [5]. The uniqueness of the injection Φ of R_+ into R_+ plays an essential role in this study, of course. These endomorphisms constitute a completely, linearly ordered semi-field; and the injections of its additive semi-group into its multiplicative group are the exponential functions $\xi \to u^{\xi}$. Thus R_+ is imbedded in a completely ordered abelian group R, the endomorphisms of which constitute a completely ordered field with additive group isomorphic to R. Of course it is obvious from the foregoing considerations that two completely ordered abelian groups are necessarily isomorphic, and that two completely ordered fields are necessarily isomorphic in just one way.

In order to describe the operations upon real numbers in a more concrete form suitable for computational purposes, we return to Kolmogorov's paper. It is easy to verify that in terms of the function

$$\xi \to [\xi] = \max_k \left\{ m; \frac{m}{k} < \xi \right\}$$

we have

$$(\xi \oplus \eta)(n) = \max_k \left[\frac{\xi(kn) + \eta(kn)}{k} \right].$$

A similar formula holds also for multiplication:

$$(\xi \otimes \eta)(n) = \max_k \left[\frac{\xi(kn) \cdot \eta(kn)}{k^2} \right].$$

From the fact that $\xi = \lim_{n \to \infty} \frac{\xi(n)}{n}$, it follows at once that the set $\{n; \xi(n) = \eta(n), \xi \neq \eta\}$ is finite. Consequently if b is any natural number other than 0 or 1, the real number ξ is determined by the sequence $\xi(b^p)$. Upon investigation it is seen that properties (1) and (2) above imply $\xi(b^{p+1}) = b\xi(b^p) + \xi_{p+1}$ where ξ_{p+1} is a natural number such that $0 \leq \xi_{p+1} < b$ and that $\xi_{p+1} > 0$ for infinitely many values of p. Putting $\xi_0 = \xi(1)$ we see that with each positive real number there is associated a unique non-terminating b-adic development

$$\xi_0 \, \xi_1 \, \xi_2 \, \xi_3 \cdots \xi_{p+1} \cdots$$

from which ξ itself may be recovered analytically by the formula

$$\xi = \lim_{q \to \infty} \sum_{l=0}^{q} \frac{\xi_l}{b^l}.$$

Conversely, for arbitrary ξ_l, $l \geq 0$, where $0 \leq \xi_l < b$ for $l \geq 1$ and infinitely many ξ_l are different from 0, this formula determines a unique positive real number with the indicated b-adic development. The formulas given above for addition and multiplication can then be translated in terms of b-adic developments. This is simple only in the case of addition, as is well-known. Formulas for change of base b can also be developed, if desired.

In a systematic presentation of this theory of real numbers it would be convenient to start with a discussion of the theory of measurement. In the present context, however, there seemed to be advantages in first giving the above outline, starting with the semi-group G, and then adding some comments on ways of replacing G by some more general type of structure. The essential modification consists in restricting the addition in G as drastically as possible without disturbing the existence and isomorphic character of the injection Φ. Thus Hölder's theorem emerges unscathed, in the generalized form given by Krantz and mentioned above. In an axiomatic treatment the system S which generalizes G is most easily described by presenting it as a linearly ordered set equipped with a minimal element θ and a family of scales, where a scale with unit s is simply an injection $n \to ns$ of an initial segment of N into S such that $0s = \theta$ and $1s = s > \theta$. The axioms stating in terms of the given order for S how the different scales mesh with one another read as follows [6]:

(I) if ms, ps and $(n+q)\,t$ exist and $ms \leq nt$ and $ps \leq qt$, then $(m+p)\,s$ exists and $(m+p)\,s \leq (n+q)\,t$;

(II) if $\theta < s < t$, then there exist u and n such that $s \leq nu < (n+1)\,u \leq t$;

(III) (archidedean property) the set $\{n; ns < t\}$ is finite.

We can define a restricted binary operation $+$ in S by putting $ms + ps = (m+p)\,s$ when the latter element exists. Once Krantz's generalization of Hölder's theorem has been established, this operation is seen to be commutative and associative insofar as its restricted nature permits. When S has a minimal $u > \theta$ the injection Φ required for Krantz's theorem is determined as follows: every element of S is of the form nu, and Φ: $nu \to \frac{n}{1} \in R_+$. Otherwise we take u to be an arbitrary non-maximal element of S and put Φ: $x \to \xi_{x,u}$ where

$$\xi_{x,u} = sup\left\{\frac{m}{n}\,;\, mv < x \text{ and } u < nv \text{ for some } v \in S\right\}.$$

These results are particularly interesting when S is a semi-group germ and have very important applications to the theory of angle-measurement, the theory of rotations in Euclidean geometry and the theory of the multiplicative group of the complex field. The latter theory includes a unified treatment of the elementary functions — trigonometric, hyperbolic, and exponential — and their inverses. These applications were discussed in part by Behrend [5] and have been further elaborated by myself, with interesting analytical insights which will not be reported here.

If one should wish to teach such a theory of the real numbers as we have just outlined, what preparation would be required of one's students? First of all, a thorough command of the theory of natural numbers and their ordinal and algebraic properties would be absolutely indispensable. From set-theory comparatively little is needed beyond the basic concepts of set and function, while from abstract algebra the notions of binary operation, groupoid, and morphism are essential. A few basic concepts are also required from the theory of order. Most of the concepts mentioned can be developed through the study of the natural numbers themselves, as is well-known. Inasmuch as the theory of measurement is taken here as the point of departure for the construction of a theory of the real numbers, it is evident that the student should have adequate previous experience with measurement procedures in various geometrical and physical settings. It should be noted that the measurement of time is even more instructive here than the measurement of length or other geometrical quantities. Indeed, time has a natural psychobiological order, and time scales can be marked off by the oscillations of pendula of various lengths, all set in motion from rest at the same initial instant of time. In this particular physical situation the axioms I-III of the preceding paragraph have a rather perspicuous interpretation. The theory of measurement leads directly to quite concrete experience with infinite sequences, which should be a part of the student's preparation for the formal study of real numbers, as defined here. In fact, it would probably be desirable for this preparation to include the usual intuitive or heuristic introduction to calculating with decimal developments and perhaps dyadic developments as well. In my opinion, the sort of preparation suggested here is quite compatible with the more advanced modern programs of school mathematics, and could be completed early enough to permit the inclusion of the present theory of real numbers in the last two years of the secondary school program.

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