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Artikel: SOME CONVERSE THEOREMS ON THE ABSCISSAE OF SUMMABILITY OF GENERAL DIRICHLET SERIES

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Kapitel: Concluding Remarks

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From (4.6) and (4.7) we obtain, appealing first to Theorem A and then to Lemma 3,

$$A(x) = o(x^{\alpha+1+\delta}), \; A^k(x) = o(x^{k+\alpha+1+\delta}), \; 0 \leqslant k < r. \qquad (4.8)$$

Now Lemma 2 establishes the summability (R, l_n, k) of $\Sigma a_n l_n^{-(\alpha+1+\delta)}$, or, of $\Sigma a_n l_n^{-s}$ for $\sigma \geqslant \alpha+1+\delta$ with arbitrary $\delta>0$. Hence $\sigma_k \leqslant \alpha+1$ as required.

(B) We now choose γ such that $(\alpha+1 \leqslant)\, \sigma_r < \gamma$ and note that $\alpha+1+\delta$ can be replaced by γ in (4.7) and (4.8), so that, arguing as before, we establish the summability (R, l_n, k), $0 \leqslant k < r$, of $\Sigma a_n l_n^{-\gamma}$ where $\gamma > \sigma_r$ is arbitrary. Hence $\sigma_k \leqslant \sigma_r$ while $\sigma_r \leqslant \sigma_k$ universally, i.e., $\sigma_k = \sigma_r$ as we wished to prove.

Deduction 3. *If, for the Dirichlet series* $\Sigma a_n l_n^{-s}$, $\sigma_r > -\infty$ *and* $\underline{\lim}\, l_n/l_{n-1} > 1$, *then* $\sigma_k = \sigma_r$ *for* $0 \leqslant k < r$.

Proof. The hypothesis $\underline{\lim}\, l_n/l_{n-1} > 1$ makes

$$a_{n+1} + a_{n+2} + \ldots + a_m = 0 \quad \text{for} \quad l_n < l_m < l_n + \varepsilon l_n$$

if ε is sufficiently small and $n > n_0(\varepsilon)$. Hence, for any ρ, in particular, for $\rho \leqslant \sigma_r$,

$$\overline{\lim_{n\to\infty}} \max_{l_n \leqq l_m < l_n + \varepsilon l_n} \frac{|a_{n+1} + a_{n+2} + \ldots + a_m|}{l_n^\rho} = o(1), \; \varepsilon \to 0 .$$

The desired conclusion now follows from Theorem I (B) with alternative (2.4) (b).

In the above proof we have supposed that $\sigma_r < \infty$, the case $\sigma_r = \infty$ being trivial.

Concluding Remarks

A few remarks are offered in conclusion, supplementing some made in the beginning. Though Theorem A in one form is Karamata's (as already said), a particularization of it ([12], Corollary VI with Tauberian *O*-condition) is a much older theorem of Ananda-Rau's ([1], Theorem 16; [2], Theorem 4). Ananda Rau left open one case of his theorem which Bosanquet ([4], Theorems 2, 3), Minakshisundaram and Rajagopal ([10], Theorem 1 and Corollaries 1.1, 1.3; [11], Theorem A and Corollaries A_1, A_2) have independently settled, even for some extensions of Ananda Rau's theorem. The theorem mentioned at the outset as being due to Chandrasekharan and Minakshisundaram ([6], p. 21, Theorem 1.82) is, in fact, a further extension of one of the extensions of Ananda Rau's theorem given by

Bosanquet ([4], Theorem 3). In the present context, it is rather less effective than the completely independent two-fold result of Karamata's in the same direction ([9], Théorèmes 1a), 3f)), reformulated as Theorem A. That is to say, precisely, Theorem A gives rise to a basic converse theorem on abscissae of summability of general Dirichlet series (Theorem I of this paper) which is more natural and suggestive as well as more comprehensive than the like basic theorem resulting from the line of development followed by Chandrasekharan and Minakshisundaram ([6], p. 86, Theorem 3.71). [1])

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1) Indeed the Chandrasekharan-Minakshisundaram theorem just referred to is deducible from Theorem I, its case $\sigma_r < \alpha + \mu$ [or, case $\sigma_r \geqq \alpha + \mu$] from part (A) [or, part (B)] of Theorem I with hypothesis (2.2) (b) and $x^{\rho} = x^{\alpha} \{ \theta(x) \}^{\mu}$, $\theta(x) = x^{(r-\alpha+\gamma)/(r+\mu)}$, $\sigma_r < \gamma < \alpha + \mu$ [or, hypothesis (2.4) (b) and $x^{\rho} = x^{\alpha+\mu}$].