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From (4.6) and (4.7) we obtain, appealing first to Theorem A and then to Lemma 3,

$$A(x) = o(x^{\alpha+1+\delta}), \quad A^k(x) = o(x^{k+\alpha+1+\delta}), \quad 0 \leq k < r. \quad (4.8)$$

Now Lemma 2 establishes the summability (R, l_n, k) of $\sum a_n l_n^{-(\alpha+1+\delta)}$, or, of $\sum a_n l_n^{-s}$ for $\sigma \geq \alpha+1+\delta$ with arbitrary $\delta > 0$. Hence $\sigma_k \leq \alpha+1$ as required.

(B) We now choose γ such that $(\alpha+1 \leq) \sigma_r < \gamma$ and note that $\alpha+1+\delta$ can be replaced by γ in (4.7) and (4.8), so that, arguing as before, we establish the summability (R, l_n, k) , $0 \leq k < r$, of $\sum a_n l_n^{-\gamma}$ where $\gamma > \sigma_r$ is arbitrary. Hence $\sigma_k \leq \sigma_r$ while $\sigma_r \leq \sigma_k$ universally, i.e., $\sigma_k = \sigma_r$ as we wished to prove.

DEDUCTION 3. *If, for the Dirichlet series $\sum a_n l_n^{-s}$, $\sigma_r > -\infty$ and $\lim l_n/l_{n-1} > 1$, then $\sigma_k = \sigma_r$ for $0 \leq k < r$.*

Proof. The hypothesis $\lim l_n/l_{n-1} > 1$ makes

$$a_{n+1} + a_{n+2} + \dots + a_m = 0 \quad \text{for } l_n < l_m < l_n + \varepsilon l_n$$

if ε is sufficiently small and $n > n_0(\varepsilon)$. Hence, for any ρ , in particular, for $\rho \leq \sigma_r$,

$$\overline{\lim}_{n \rightarrow \infty} \max_{l_n \leq l_m < l_n + \varepsilon l_n} \frac{|a_{n+1} + a_{n+2} + \dots + a_m|}{l_n^\rho} = o(1), \quad \varepsilon \rightarrow 0.$$

The desired conclusion now follows from Theorem I (B) with alternative (2.4) (b).

In the above proof we have supposed that $\sigma_r < \infty$, the case $\sigma_r = \infty$ being trivial.

CONCLUDING REMARKS

A few remarks are offered in conclusion, supplementing some made in the beginning. Though Theorem A in one form is Karamata's (as already said), a particularization of it ([12], Corollary VI with Tauberian O -condition) is a much older theorem of Ananda-Rau's ([1], Theorem 16; [2], Theorem 4). Ananda Rau left open one case of his theorem which Bosanquet ([4], Theorems 2, 3), Minakshisundaram and Rajagopal ([10], Theorem 1 and Corollaries 1.1, 1.3; [11], Theorem A and Corollaries A₁, A₂) have independently settled, even for some extensions of Ananda Rau's theorem. The theorem mentioned at the outset as being due to Chandrasekharan and Minakshisundaram ([6], p. 21, Theorem 1.82) is, in fact, a further extension of one of the extensions of Ananda Rau's theorem given by

Bosanquet ([4], Theorem 3). In the present context, it is rather less effective than the completely independent two-fold result of Karamata's in the same direction ([9], Théorèmes 1a), 3f)), reformulated as Theorem A. That is to say, precisely, Theorem A gives rise to a basic converse theorem on abscissae of summability of general Dirichlet series (Theorem I of this paper) which is more natural and suggestive as well as more comprehensive than the like basic theorem resulting from the line of development followed by Chandrasekharan and Minakshisundaram ([6], p. 86, Theorem 3.71). ¹⁾

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REFERENCES

- [1] ANANDA-RAU, K., *On some properties of Dirichlet's series*. Smith's Prize Essay, Cambridge 1918.
- [2] —— On the convergence and summability of Dirichlet's series. *Proc. London Math. Soc.*, (2), 34 (1932), 414-440.
- [3] BOSANQUET, L. S., On the summability of Fourier series. *Proc. London Math. Soc.*, (2), 31 (1931), 144-164.
- [4] —— Note on convexity theorems. *J. London Math. Soc.*, 18 (1943), 239-248.
- [5] —— Note on the converse of Abel's theorem. *J. London Math. Soc.*, 19 (1944), 161-168.
- [6] CHANDRASEKHARAN, K. and S. MINAKSHISUNDARAM, *Typical Means* (Tata Institute of Fundamental Research Monographs in Mathematics and Physics, No. 1), Bombay 1952.
- [7] GANAPATHY IYER, V., Tauberian and summability theorems on Dirichlet's series. *Ann. of Math.*, 36 (1935), 100-116.
- [8] KARAMATA, J., On an inversion of Cesàro's method of summing divergent series (Serbian). *Glas. Srpske Akad. Nauk*, 191 (1948), 1-37.
- [9] —— Quelques théorèmes inverses relatifs aux procédés de sommabilité de Cesàro et Riesz. *Acad. Serbe Sci. Publ. Inst. Math.*, 3 (1950), 53-71.
- [10] MINAKSHISUNDARAM, S. and C. T. RAJAGOPAL, An extension of a Tauberian theorem of L. J. Mordell. *Proc. London Math. Soc.*, (2), 50 (1945), 242-255.
- [11] —— and C. T. RAJAGOPAL, On a Tauberian theorem of K. Ananda Rau. *Quart. J. Math. Oxford Ser.*, 17 (1946), 153-161.
- [12] RAJAGOPAL, C. T., On Tauberian theorems for the Riemann-Liouville integral. *Acad. Serbe Sci. Publ. Inst. Math.*, 6 (1954), 27-46.

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¹⁾ Indeed the Chandrasekharan-Minakshisundaram theorem just referred to is deducible from Theorem I, its case $\sigma_r < \alpha + \mu$ [or, case $\sigma_r \geq \alpha + \mu$] from part (A) [or, part (B)] of Theorem I with hypothesis (2.2) (b) and $x^\rho = x^\alpha \{0(x)\}^\mu$, $0(x) = x^{(r-\alpha+\gamma)/(r+\mu)}$, $\sigma_r < \gamma < \alpha + \mu$ [or, hypothesis (2.4) (b) and $x^\rho = x^{\alpha+\mu}$].