Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 15 (1969)

Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SOME CONVERSE THEOREMS ON THE ABSCISSAE OF SUMMABILITY OF GENERAL DIRICHLET SERIES

Autor: Rajagopal, C. T.

Kapitel: Concluding Remarks

DOI: https://doi.org/10.5169/seals-43225

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 01.02.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

From (4.6) and (4.7) we obtain, appealing first to Theorem A and then to Lemma 3,

$$A(x) = o(x^{\alpha+1+\delta}), \quad A^k(x) = o(x^{k+\alpha+1+\delta}), \quad 0 \leqslant k < r. \tag{4.8}$$

Now Lemma 2 establishes the summability (R, l_n, k) of $\Sigma a_n l_n^{-(\alpha+1+\delta)}$, or, of $\Sigma a_n l_n^{-s}$ for $\sigma \geqslant \alpha+1+\delta$ with arbitrary $\delta > 0$. Hence $\sigma_k \leqslant \alpha+1$ as required.

(B) We now choose γ such that $(\alpha+1 \leqslant)\, \sigma_r < \gamma$ and note that $\alpha+1+\delta$ can be replaced by γ in (4.7) and (4.8), so that, arguing as before, we establish the summability (R, l_n, k), $0 \leqslant k < r$, of $\Sigma a_n l_n^{-\gamma}$ where $\gamma > \sigma_r$ is arbitrary. Hence $\sigma_k \leqslant \sigma_r$ while $\sigma_r \leqslant \sigma_k$ universally, i.e., $\sigma_k = \sigma_r$ as we wished to prove.

DEDUCTION 3. *If, for the Dirichlet series* $\Sigma a_n l_n^{-s}$, $\sigma_r > -\infty$ *and* $\underline{\lim}\, l_n/l_{n-1} > 1$, *then* $\sigma_k = \sigma_r$ *for* $0 \leqslant k < r$.

Proof. The hypothesis $\underline{\lim}\, l_n/l_{n-1} > 1$ makes

$$a_{n+1} + a_{n+2} + \ldots + a_m = 0 \quad \text{for} \quad l_n < l_m < l_n + \varepsilon l_n$$

if ε is sufficiently small and $n > n_0(\varepsilon)$. Hence, for any ρ, in particular, for $\rho \leqslant \sigma_r$,

$$\overline{\lim_{n\to\infty}} \max_{l_n \leq l_m < l_n + \varepsilon l_n} \frac{|a_{n+1} + a_{n+2} + \ldots + a_m|}{l_n^{\rho}} = o(1), \quad \varepsilon \to 0.$$

The desired conclusion now follows from Theorem I (B) with alternative (2.4) (b).

In the above proof we have supposed that $\sigma_r < \infty$, the case $\sigma_r = \infty$ being trivial.

CONCLUDING REMARKS

A few remarks are offered in conclusion, supplementing some made in the beginning. Though Theorem A in one form is Karamata's (as already said), a particularization of it ([12], Corollary VI with Tauberian O-condition) is a much older theorem of Ananda-Rau's ([1], Theorem 16; [2], Theorem 4). Ananda Rau left open one case of his theorem which Bosanquet ([4], Theorems 2, 3), Minakshisundaram and Rajagopal ([10], Theorem 1 and Corollaries 1.1, 1.3; [11], Theorem A and Corollaries A_1, A_2) have independently settled, even for some extensions of Ananda Rau's theorem. The theorem mentioned at the outset as being due to Chandrasekharan and Minakshisundaram ([6], p. 21, Theorem 1.82) is, in fact, a further extension of one of the extensions of Ananda Rau's theorem given by

Bosanquet ([4], Theorem 3). In the present context, it is rather less effective than the completely independent two-fold result of Karamata's in the same direction ([9], Théorèmes 1a), 3f)), reformulated as Theorem A. That is to say, precisely, Theorem A gives rise to a basic converse theorem on abscissae of summability of general Dirichlet series (Theorem I of this paper) which is more natural and suggestive as well as more comprehensive than the like basic theorem resulting from the line of development followed by Chandrasekharan and Minakshisundaram ([6], p. 86, Theorem 3.71). [1])

I am indebted to Prof. Bosanquet for some very useful remarks on the original version of this paper which have led to the preparation of the present version.

REFERENCES

[1] ANANDA-RAU, K., *On some properties of Dirichlet's series.* Smith's Prize Essay, Cambridge 1918.

[2] —— On the convergence and summability of Dirichlet's series. *Proc. London Math. Soc.*, (2), 34 (1932), 414-440.

[3] BOSANQUET, L. S., On the summability of Fourier series. *Proc. London Math. Soc.*, (2), 31 (1931), 144-164.

[4] —— Note on convexity theorems. *J. London Math. Soc.*, 18 (1943), 239-248.

[5] —— Note on the converse of Abel's theorem. *J. London Math. Soc.*, 19 (1944), 161-168.

[6] CHANDRASEKHARAN, K. and S. MINAKSHISUNDARAM, *Typical Means* (Tata Institute of Fundamental Research Monographs in Mathematics and Physics, No. 1), Bombay 1952.

[7] GANAPATHY IYER, V., Tauberian and summability theorems on Dirichlet's series. *Ann. of Math.*, 36 (1935), 100-116.

[8] KARAMATA, J., On an inversion of Cesàro's method of summing divergent series (Serbian). *Glas. Srpske Akad. Nauk*, 191 (1948), 1-37.

[9] —— Quelques théorèmes inverses relatifs aux procédés de sommabilité de Cesàro et Riesz. *Acad. Serbe Sci. Publ. Inst. Math.*, 3 (1950), 53-71.

[10] MINAKSHISUNDARAM, S. and C. T. RAJAGOPAL, An extension of a Tauberian theorem of L. J. Mordell. *Proc. London Math. Soc.*, (2), 50 (1945), 242-255.

[11] —— and C. T. RAJAGOPAL, On a Tauberian theorem of K. Ananda Rau. *Quart. J. Math. Oxford Ser.*, 17 (1946), 153-161.

[12] RAJAGOPAL, C. T., On Tauberian theorems for the Riemann-Liouville integral. *Acad. Serbe Sci. Publ. Inst. Math.*, 6 (1954), 27-46.

(Reçu le 15 Juillet 1968)

The Ramanujan Institute
University of Madras
Madras-5, India.

---

[1]) Indeed the Chandrasekharan-Minakshisundaram theorem just referred to is deducible from Theorem I, its case $\sigma_r < \alpha + \mu$ [or, case $\sigma_r \geqq \alpha + \mu$] from part (A) [or, part (B)] of Theorem I with hypothesis (2.2) (b) and $x^{\rho} = x^{\alpha} \{ \theta(x) \}^{\mu}$, $\theta(x) = x^{(r-\alpha+\gamma)/(r+\mu)}$, $\sigma_r < \gamma < \alpha + \mu$ [or, hypothesis (2.4) (b) and $x^{\rho} = x^{\alpha+\mu}$].