

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 15 (1969)
Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SOME CONVERSE THEOREMS ON THE ABSCISSAE OF
SUMMABILITY OF GENERAL DIRICHLET SERIES
Autor: Rajagopal, C. T.
Kapitel: §4. Further applications
DOI: <https://doi.org/10.5169/seals-43225>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 11.01.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

It may be observed that the assumption $\alpha+1+p^{-1} \geq 0$ involves no loss of generality since $\alpha+1+p^{-1} < 0$ makes successively $a_n + |a_n| \equiv 0$, $a_n \equiv 0$ and so $\sigma_r = -\infty$ for all $r \geq 0$.

THEOREM V. *In Theorem II, let hypothesis (i) be omitted on account of its being implicit (with $q = 0$, $\rho = \alpha+1$) in hypothesis (iii) modified as under. Let hypothesis (ii) be retained with ρ changed to $\alpha+1$, and hypothesis (iii) replaced by*

$$a_n = O[l_n^\alpha(l_n - l_{n-1})]. \quad (3.11)$$

Then the conclusion is that $\Sigma a_n l_n^{-s}$ is summable (R, l_n, k) , $0 \leq k < r$, for σ satisfying (3.2).

THEOREM VI. *If, in Theorem V, (3.11) alone is changed to*

$$\sum_{v=1}^n |a_v|^p l_v^p (l_v - l_{v-1})^{1-p} = O[l_n^{p(\alpha+1)+1}], \quad p > 1, \quad \alpha + 1 + p^{-1} \geq 0,$$

the conclusion will become the assertion that $\Sigma a_n l_n^{-s}$ is summable (R, l_n, k) , $0 \leq k < r$, for σ satisfying (3.8).

The proofs of Theorems V, VI are omitted, being obvious simplifications of those of Theorems III, IV, involving the use of Theorem I (A) with hypothesis (2.2) (b) instead of (2.2.) (a) as formerly. Theorems V and VI, as pointed out by Chandrasekharan and Minakshisundaram, yield Ananda Rau's and Ganapathy Iyer's extensions of the Schnee-Landau theorem when $\alpha \rightarrow +0$.

§ 4. FURTHER APPLICATIONS

Theorem I (A) is a base which, combined with Theorem B, produces Theorem II, and in this sense Theorem I (A) may be said to correspond to Theorem II. There are results corresponding to each of Theorems III-VI in the same sense. For instance, Deduction 1 below corresponds to Theorem III and shows how other deductions corresponding to Theorems IV-VI may be formulated. Deductions 2,3 are further examples of results based on Theorem I.

DEDUCTION 1. (A) *In Theorem I (A), suppose that $\sigma_r < \alpha+1$ and that (2.2) (a) is replaced by*

$$a_n = O_R[l_n^\alpha(l_n - l_{n-1})], \quad l_n - l_{n-1} = O(l_n^{(r-\alpha+\sigma_r)/(r+1)}). \quad (4.1)$$

Then

$$\sigma_k \leq \frac{(r-k)(\alpha+1) + (k+1)\sigma_r}{r+1} \quad (0 \leq k < r). \quad (4.2)$$

(B) In Theorem I (B), suppose that $\sigma_r \geq \alpha+1$ and that (2.4) (a) is replaced by

$$a_n = O_R[l_n^\alpha(l_n - l_{n-1})], \quad l_n - l_{n-1} = o(l_n). \quad (4.3)$$

Then

$$\sigma_k = \sigma_r \quad (0 \leq k < r). \quad (4.4)$$

Proof. The proof of part (A) is on the lines of that of Theorem III excepting that now there is no appeal to Theorem B. The proof of part (B) may need a further explanation as follows. The two conditions of (4.3) together imply $a_n = o_R(l_n^{\alpha+1})$ which, along with the first condition of (4.3), readily gives us

$$\lim_{n \rightarrow \infty} \max_{l_n \leq l_m < l_n + \varepsilon l_n} \frac{a_n + a_{n+1} + \dots + a_m}{l_n^{\alpha+1}} = o_R(1), \quad \varepsilon \rightarrow 0.$$

The conclusion (4.4) now follows obviously from Theorem I (B) with alternative (2.4) (a) and $\rho = \alpha+1$.

The following deduction supplements the preceding and has been kindly suggested by Prof. Bosanquet.

DEDUCTION 2. Suppose that, in Deduction 1, we replace (4.1) in (A) and (4.3) in (B) by the common hypothesis

$$a_n = O_R[l_n^\alpha(l_n - l_{n-1})], \quad \sigma_r \geq 0. \quad (4.5)$$

Then we have, for $0 \leq k < r$, EITHER (A) $\sigma_k \leq \alpha+1$, OR (B) $\sigma_k = \sigma_r$, according as $\sigma_r < \alpha+1$ or $\sigma_r \geq \alpha+1$.

Proof. (A) We choose γ such that $(0 \leq) \sigma_r < \gamma < \alpha+1$ and, as in (2.6), assume that $B^r(x) = o(x^r)$. Then we infer, from an application of Lemma 1,

$$A^r(x) = o(x^{r+\gamma}) = o(x^{r+\alpha+1+\delta}) \text{ for every } \delta > 0. \quad (4.6)$$

On the other hand, our hypothesis on a_n gives us first $a_n = O_R(l_n^{\alpha+1}) = o_R(l_n^{\alpha+1+\delta})$ and then, as in the proof of part (B) of Deduction 1,

$$\lim_{n \rightarrow \infty} \max_{l_n \leq l_m < l_n + \varepsilon l_n} \frac{a_n + a_{n+1} + \dots + a_m}{l_n^{\alpha+1+\delta}} = o_R(1), \quad \varepsilon \rightarrow 0. \quad (4.7)$$

From (4.6) and (4.7) we obtain, appealing first to Theorem A and then to Lemma 3,

$$A(x) = o(x^{\alpha+1+\delta}), \quad A^k(x) = o(x^{k+\alpha+1+\delta}), \quad 0 \leq k < r. \quad (4.8)$$

Now Lemma 2 establishes the summability (R, l_n, k) of $\Sigma a_n l_n^{-(\alpha+1+\delta)}$, or, of $\Sigma a_n l_n^{-s}$ for $s \geq \alpha+1+\delta$ with arbitrary $\delta > 0$. Hence $\sigma_k \leq \alpha+1$ as required.

(B) We now choose γ such that $(\alpha+1 \leq) \sigma_r < \gamma$ and note that $\alpha+1+\delta$ can be replaced by γ in (4.7) and (4.8), so that, arguing as before, we establish the summability (R, l_n, k) , $0 \leq k < r$, of $\Sigma a_n l_n^{-\gamma}$ where $\gamma > \sigma_r$ is arbitrary. Hence $\sigma_k \leq \sigma_r$ while $\sigma_r \leq \sigma_k$ universally, i.e., $\sigma_k = \sigma_r$ as we wished to prove.

DEDUCTION 3. *If, for the Dirichlet series $\Sigma a_n l_n^{-s}$, $\sigma_r > -\infty$ and $\lim l_n/l_{n-1} > 1$, then $\sigma_k = \sigma_r$ for $0 \leq k < r$.*

Proof. The hypothesis $\lim l_n/l_{n-1} > 1$ makes

$$a_{n+1} + a_{n+2} + \dots + a_m = 0 \quad \text{for} \quad l_n < l_m < l_n + \varepsilon l_n$$

if ε is sufficiently small and $n > n_0(\varepsilon)$. Hence, for any ρ , in particular, for $\rho \leq \sigma_r$,

$$\lim_{n \rightarrow \infty} \max_{l_n \leq l_m < l_n + \varepsilon l_n} \frac{|a_{n+1} + a_{n+2} + \dots + a_m|}{l_n^\rho} = o(1), \quad \varepsilon \rightarrow 0.$$

The desired conclusion now follows from Theorem I (B) with alternative (2.4) (b).

In the above proof we have supposed that $\sigma_r < \infty$, the case $\sigma_r = \infty$ being trivial.

CONCLUDING REMARKS

A few remarks are offered in conclusion, supplementing some made in the beginning. Though Theorem A in one form is Karamata's (as already said), a particularization of it ([12], Corollary VI with Tauberian O -condition) is a much older theorem of Ananda-Rau's ([1], Theorem 16; [2], Theorem 4). Ananda Rau left open one case of his theorem which Bosanquet ([4], Theorems 2, 3), Minakshisundaram and Rajagopal ([10], Theorem 1 and Corollaries 1.1, 1.3; [11], Theorem A and Corollaries A_1, A_2) have independently settled, even for some extensions of Ananda Rau's theorem. The theorem mentioned at the outset as being due to Chandrasekharan and Minakshisundaram ([6], p. 21, Theorem 1.82) is, in fact, a further extension of one of the extensions of Ananda Rau's theorem given by