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Artikel: SOME CONVERSE THEOREMS ON THE ABSCISSAE OF SUMMABILITY OF GENERAL DIRICHLET SERIES

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It may be observed that the assumption $\alpha+1+p^{-1}\geqslant 0$ involves no loss of generality since $\alpha+1+p^{-1}<0$ makes successively $a_n+|a_n|\equiv 0$, $a_n\equiv 0$ and so $\sigma_r = -\infty$ for all $r\geqslant 0$.

THEOREM V. *In Theorem* II, *let hypothesis* (i) *be omitted on account of its being implicit* (*with* $q=0$, $\rho=\alpha+1$) *in hypothesis* (iii) *modified as under. Let hypothesis* (ii) *be retained with* ρ *changed to* $\alpha+1$, *and hypothesis* (iii) *replaced by*

$$a_n = O\left[l_n^{\alpha}(l_n-l_{n-1})\right]. \tag{3.11}$$

Then the conclusion is that $\Sigma a_n\, l_n^{-s}$ *is summable* (R, l_n, k), $0\leqslant k<r$, *for* σ *satisfying* (3.2).

THEOREM VI. *If, in Theorem* V, (3.11) *alone is changed to*

$$\sum_{\nu=1}^{n} |a_\nu|^p\, l_\nu^p (l_\nu - l_{\nu-1})^{1-p} = O\left[l_n^{p(\alpha+1)+1}\right],\ p>1,\ \alpha+1+p^{-1}\geqslant 0,$$

the conclusion will become the assertion that $\Sigma a_n\, l_n^{-s}$ *is summable* (R, l_n, k), $0\leqslant k<r$, *for* σ *satisfying* (3.8).

The proofs of Theorems V, VI are omitted, being obvious simplifications of those of Theorems III, IV, involving the use of Theorem I (A) with hypothesis (2.2) (b) instead of (2.2.) (a) as formerly. Theorems V and VI, as pointed out by Chandrasekharan and Minakshisundaram, yield Ananda Rau's and Ganapathy Iyer's extensions of the Schnee-Landau theorem when $\alpha\to+0$.

§ 4. FURTHER APPLICATIONS

Theorem I (A) is a base which, combined with Theorem B, produces Theorem II, and in this sense Theorem I (A) may be said to correspond to Theorem II. There are results corresponding to each of Theorems III-VI in the same sense. For instance, Deduction 1 below corresponds to Theorem III and shows how other deductions corresponding to Theorems IV-VI may be formulated. Deductions 2,3 are further examples of results based on Theorem I.

DEDUCTION 1. (A) *In Theorem* I (A), *suppose that* $\sigma_r<\alpha+1$ *and that* (2.2) (a) *is replaced by*

$$a_n = O_R\left[l_n^{\alpha}(l_n-l_{n-1})\right],\ l_n - l_{n-1} = O\left(l_n^{(r-\alpha+\sigma_r)/(r+1)}\right). \tag{4.1}$$

Then

$$\sigma_k \leqslant \frac{(r-k)(\alpha+1) + (k+1)\sigma_r}{r+1} \quad (0 \leqslant k < r). \tag{4.2}$$

(B) *In Theorem* I (B), *suppose that* $\sigma_r \geqslant \alpha+1$ *and that* (2.4) (a) *is replaced by*

$$a_n = O_R[l_n^\alpha (l_n - l_{n-1})], \; l_n - l_{n-1} = o(l_n). \tag{4.3}$$

Then

$$\sigma_k = \sigma_r \; (0 \leqslant k < r). \tag{4.4}$$

Proof. The proof of part (A) is on the lines of that of Theorem III excepting that now there is no appeal to Theorem B. The proof of part (B) may need a further explanation as follows. The two conditions of (4.3) together imply $a_n = o_R(l^{\alpha+1}_n)$ which, along with the first condition of (4.3), readily gives us

$$\overline{\lim_{n\to\infty}} \max_{l_n \leqq l_m < l_n + \varepsilon l_n} \frac{a_n + a_{n+1} + \ldots + a_m}{l_n^{\alpha+1}} = o_R(1), \; \varepsilon \to 0.$$

The conclusion (4.4) now follows obviously from Theorem I (B) with alternative (2.4) (a) and $\rho = \alpha+1$.

The following deduction supplements the preceding and has been kindly suggested by Prof. Bosanquet.

DEDUCTION 2. *Suppose that, in Deduction* 1, *we replace* (4.1) *in* (A) *and* (4.3) *in* (B) *by the common hypothesis*

$$a_n = O_R[l_n^\alpha (l_n - l_{n-1})], \; \sigma_r \geqslant 0. \tag{4.5}$$

Then we have, for $0 \leqslant k < r$, EITHER (A) $\sigma_k \leqslant \alpha+1$, OR (B) $\sigma_k = \sigma_r$, *according as* $\sigma_r < \alpha+1$ *or* $\sigma_r \geqslant \alpha+1$.

Proof. (A) We choose γ such that $(0 \leqslant)\, \sigma_r < \gamma < \alpha+1$ and, as in (2.6), assume that $B^r(x) = o(x^r)$. Then we infer, from an application of Lemma 1,

$$A^r(x) = o(x^{r+\gamma}) = o(x^{r+\alpha+1+\delta}) \text{ for every } \delta > 0. \tag{4.6}$$

On the other hand, our hypothesis on a_n gives us first $a_n = O_R(l_n^{\alpha+1}) = o_R(l_n^{\alpha+1+\delta})$ and then, as in the proof of part (B) of Deduction 1,

$$\overline{\lim_{n\to\infty}} \max_{l_n \leqq l_m < l_n + \varepsilon l_n} \frac{a_n + a_{n+1} + \ldots + a_m}{l_n^{\alpha+1+\delta}} = o_R(1), \; \varepsilon \to 0. \tag{4.7}$$

From (4.6) and (4.7) we obtain, appealing first to Theorem A and then to Lemma 3,

$$A(x) = o(x^{\alpha+1+\delta}), \quad A^k(x) = o(x^{k+\alpha+1+\delta}), \quad 0 \leqslant k < r. \tag{4.8}$$

Now Lemma 2 establishes the summability (R, l_n, k) of $\Sigma a_n l_n^{-(\alpha+1+\delta)}$, or, of $\Sigma a_n l_n^{-s}$ for $\sigma \geqslant \alpha+1+\delta$ with arbitrary $\delta>0$. Hence $\sigma_k \leqslant \alpha+1$ as required.

(B) We now choose γ such that $(\alpha+1 \leqslant)\, \sigma_r < \gamma$ and note that $\alpha+1+\delta$ can be replaced by γ in (4.7) and (4.8), so that, arguing as before, we establish the summability (R, l_n, k), $0 \leqslant k < r$, of $\Sigma a_n l_n^{-\gamma}$ where $\gamma > \sigma_r$ is arbitrary. Hence $\sigma_k \leqslant \sigma_r$ while $\sigma_r \leqslant \sigma_k$ universally, i.e., $\sigma_k = \sigma_r$ as we wished to prove.

Deduction 3. *If, for the Dirichlet series* $\Sigma a_n l_n^{-s}$, $\sigma_r > -\infty$ *and* $\underline{\lim}\, l_n / l_{n-1} > 1$, *then* $\sigma_k = \sigma_r$ *for* $0 \leqslant k < r$.

Proof. The hypothesis $\underline{\lim}\, l_n / l_{n-1} > 1$ makes

$$a_{n+1} + a_{n+2} + \ldots + a_m = 0 \quad \text{for} \quad l_n < l_m < l_n + \varepsilon l_n$$

if ε is sufficiently small and $n > n_0(\varepsilon)$. Hence, for any ρ, in particular, for $\rho \leqslant \sigma_r$,

$$\overline{\lim_{n\to\infty}} \max_{l_n \leqq l_m < l_n + \varepsilon l_n} \frac{|a_{n+1} + a_{n+2} + \ldots + a_m|}{l_n^{\rho}} = o(1), \quad \varepsilon \to 0.$$

The desired conclusion now follows from Theorem I (B) with alternative (2.4) (b).

In the above proof we have supposed that $\sigma_r < \infty$, the case $\sigma_r = \infty$ being trivial.

Concluding Remarks

A few remarks are offered in conclusion, supplementing some made in the beginning. Though Theorem A in one form is Karamata's (as already said), a particularization of it ([12], Corollary VI with Tauberian *O*-condition) is a much older theorem of Ananda-Rau's ([1], Theorem 16; [2], Theorem 4). Ananda Rau left open one case of his theorem which Bosanquet ([4], Theorems 2, 3), Minakshisundaram and Rajagopal ([10], Theorem 1 and Corollaries 1.1, 1.3; [11], Theorem A and Corollaries A_1, A_2) have independently settled, even for some extensions of Ananda Rau's theorem. The theorem mentioned at the outset as being due to Chandrasekharan and Minakshisundaram ([6], p. 21, Theorem 1.82) is, in fact, a further extension of one of the extensions of Ananda Rau's theorem given by