Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 15 (1969)

Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SOME CONVERSE THEOREMS ON THE ABSCISSAE OF SUMMABILITY OF GENERAL DIRICHLET SERIES

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Kapitel: §3. Applications to theorems of the Schnee-Landau type

DOI: https://doi.org/10.5169/seals-43225

exactly as (2.2) (a) implies (2.7). Since now $\gamma > \sigma_r \geqslant \rho$, the above condition in its turn implies

$$\overline{\lim_{n\to\infty}} \max_{l_n \leqq l_m < l_n + \varepsilon l_n} (b_n + b_{n+1} + \ldots + b_m) = o_R(1), \ \varepsilon \to 0 .$$

By Theorem A with hypothesis (1.2) (a) and $a = b = 0$, it follows that $\Sigma a_n l_n^{-s}$ is convergent for any σ such that $\sigma \geqslant \gamma > \sigma_r$ and therefore $\sigma_0 \leqslant \sigma_r$. But, in any case, $\sigma_0 \geqslant \sigma_k \geqslant \sigma_r$ for $0 \leqslant k < r$ and so we have the conclusion (2.5).

In the preceding argument we have supposed that $\sigma_r < \infty$ since $\sigma_r = \infty$ implies trivially $\sigma_k = \infty$.

§ 3. Applications to theorems of the Schnee-Landau type

Theorem II given next is the simplest of the theorems of the type mentioned above and it is a direct combination of Theorems I, B. Theorems V, VI are generalizations, respectively of Ananda-Rau's and Ganapathy Iyer's extensions of the Schnee-Landau theorem ([2], Theorem 9; [7], Theorem 10), as given by Chandrasekharan and Minakshisundaram ([6], pp. 88-9, Corollaries 3.73, 3.74). Theorems III, IV are apparently new counterparts of Theorems V, VI, the newness consisting in the replacement of the two-sided Tauberian conditions of the latter pair of theorems by analogous one-sided conditions suitably supplemented.

Theorem II. *Suppose that* (i) *the Dirichlet series,*

$$\sum_1^\infty \frac{a_n}{l_n^s}, \quad s = \sigma + i\tau ,$$

is summable (R, l_n, q) *for some* $q \geqslant 0$ *when* $\sigma > \rho$, (ii) *the sum-function* $f(s)$ *thus defined is regular for* $\sigma > \eta$ *when* $\eta < \rho$, *and satisfies the condition*

$$f(s) = O(|\tau|^r), \ r > 0, \quad \textit{uniformly for} \quad \sigma \geqslant \eta + \varepsilon > \eta,$$

(iii) *the coefficients* a_n *of the Dirichlet series satisfy* ONE *of the two alternatives* (a), (b) *of* (2.2), *but with* $\theta(x) \equiv x^{1-(\rho-\eta)/r}$. *Then the Dirichlet series is summable* (R, l_n, k), $0 \leqslant k < r$, *for*

$$\sigma > \frac{(r-k)\rho + k\eta}{r} .$$

Proof. By Theorem B, the Dirichlet series is summable (R, l_n, r'), $r' > r$, for $\sigma > \eta$ and hence $\sigma_{r'} \leqslant \eta < \rho$. Therefore it is evident from the proof of

Theorem I (A) ending with (2.10) that the Dirichlet series is summable (R, l_n, k), $0 \leqslant k < r'$, for

$$\sigma \geqslant \frac{(r'-k)\rho + k\eta}{r'},$$

whence the desired conclusion follows when we let $r' \to r$.

THEOREM III. *In Theorem* II, *let ρ be replaced by $\alpha+1$ in hypotheses* (i) *and* (ii)*; also let hypothesis* (iii) *be replaced by*

$$a_n = O_R\left[l_n^\alpha (l_n - l_{n-1})\right], \quad l_n - l_{n-1} = O\left(l_n^{\frac{r-\alpha+\eta}{r+1}}\right). \tag{3.1}$$

Then the conclusion is that $\Sigma a_n l_n^{-s}$, $s = \sigma + i\tau$, is summable (R, l_n, k), $0 \leqslant k < r$, *for*

$$\sigma > \frac{(r-k)(\alpha+1) + (k+1)\eta}{r+1}. \tag{3.2}$$

Proof. As in the proof of Theorem II, the series $\Sigma a_n l_n^{-s}$ is summable (R, l_n, r'), $r' > r$, for $\sigma > \eta$ where now $\eta < \alpha+1$, so that $\sigma_{r'} \leqslant \eta < \alpha + 1$. We begin by choosing γ and correspondingly $\theta(x)$ as follows:

$$\eta < \gamma < \alpha + 1, \quad \theta(x) \equiv x^{(r'-\alpha+\gamma)/(r'+1)}. \tag{3.3}$$

Then, since $r' > r$ and $\gamma > \eta$, we have

$$\frac{r' - \alpha + \gamma}{r' + 1} > \frac{r - \alpha + \gamma}{r + 1} > \frac{r - \alpha + \eta}{r + 1}.$$

And so (3.1) gives us, as $n \to \infty$,

$$a_n = O_R\left[l_n^\alpha l_n^{\frac{r-\alpha+\eta}{r+1}}\right] = o_R\left[l_n^\alpha l_n^{\frac{r'-\alpha+\gamma}{r'+1}}\right] = o_R\left[l_n^\alpha \theta(l_n)\right]. \tag{3.4}$$

Also, if $l_n \leqslant l_m < l_n + \varepsilon\theta(l_n)$, (3.1) again gives us as $n \to \infty$,

$$a_{n+1} + a_{n+2} + \ldots + a_m = \begin{cases} O_R\left[l_m^\alpha (l_m - l_n)\right] \text{ if } \alpha \geqslant 0, \\ O_R\left[l_n^\alpha (l_m - l_n)\right] \text{ if } \alpha < 0, \end{cases}$$

so that, whether $\alpha \geqslant 0$ or $\alpha < 0$,

$$a_{n+1} + a_{n+2} + \ldots + a_m = O_R\left[l_n^\alpha \varepsilon\, \theta(l_n)\right]. \tag{3.5}$$

In (3.4) and (3.5),

$$l_n^{\alpha}\,\theta(l_n) = l_n^{\rho'} \quad \text{where} \quad \rho' = \alpha + \frac{r' - \alpha + \gamma}{r' + 1}\ (>\gamma).$$

Hence, combining (3.4) and (3.5), we get

$$\overline{\lim_{n\to\infty}} \max_{l_n \leq l_m < l_n + \varepsilon\theta(l_n)} \frac{a_n + a_{n+1} + \dots + a_m}{l_n^{\rho'}} = o_R(1), \quad \varepsilon \to 0. \tag{3.6}$$

(3.6) and the fact, following from Theorem B, that $\Sigma a_n l_n^{-s}$ is summable (R, l_n, r'), enables us to use (2.10) in the proof of Theorem I (A) with r, ρ replaced by r', ρ' respectively, so as to infer that $\Sigma a_n l_n^{-s}$ is summable (R, l_n, k), $0 \leqslant k < r'$, for

$$\sigma \geqslant \frac{(r'-k)\,\rho' + k\gamma}{r'} = \frac{(r'-k)(\alpha+1) + (k+1)\gamma}{r'+1}.$$

This yields (3.2) as required when we let $r' \to r$ and recall that $\gamma\ (>\eta)$ can be taken arbitrarily close to η.

THEOREM IV. *In Theorem* III, (3.1) *alone can be changed to*

$$\left.\begin{aligned} &\sum_{\nu=1}^{n} (a_\nu + |a_\nu|)\, l_\nu^{p}\, (l_\nu - l_{\nu-1})^{1-p} = O\,(l_n^{p(\alpha+1)+1})^{1}), \quad l_n - l_{n-1} = \\ &= O\left[l_n^{\frac{r-\alpha-p^{-1}+\eta}{r+1-p^{-1}}} \right], \quad p > 1, \quad \alpha + 1 + p^{-1} \geqslant 0, \end{aligned}\right\} \tag{3.7}$$

with the conclusion changed in consequence to the assertion that $\Sigma a_n l_n^{-s}$ *is summable* (R, l_n, k), $0 \leqslant k < r$, *for*

$$\sigma > \frac{(r-k)(\alpha+1) + (k+1-p^{-1})\,\eta}{r+1-p^{-1}}. \tag{3.8}$$

Proof. We observe that Theorem III may be viewed as the limiting case $p = \infty$ of Theorem IV.

The proof itself is similar to that of Theorem III with the difference that the choice of γ and $\theta(x)$ in (3.3) is now altered as below:

$$\eta < \gamma < \alpha + 1, \quad \theta(x) \equiv x^{(r'-\alpha-p^{-1}+\gamma)/(r'+1-p^{-1})}$$

[1]) We suppose that $l_0 = 0$.

And furthermore the step corresponding to (3.6) is obtained as follows. Writing $1-1/p = 1/p'$, we get, for $l_n \leqslant l_m < l_n + \varepsilon\theta(l_n)$,

$$a_{n+1} + a_{n+2} + \dots + a_m \leqslant a_{n+1} + |a_{n+1}| + \dots + a_m + |a_m|$$

$$= \sum_{\nu=1}^{m-n} (a_{\nu+n} + |a_{\nu+n}|)\, l_{\nu+n} (l_{\nu+n} - l_{\nu+n-1})^{(1-p)/p} \times$$

$$\times \frac{(l_{\nu+n} - l_{\nu+n-1})^{1/p'}}{l_{\nu+n}}$$

$$\leqslant \left[\sum_{\nu=1}^{m-n} (a_{\nu+n} + |a_{\nu+n}|)^p\, l_{\nu+n}^p (l_{\nu+n} - l_{\nu+n-1})^{1-p} \right]^{1/p} \times$$

$$\times \left[\sum_{\nu=1}^{m-n} \frac{l_{\nu+n} - l_{\nu+n-1}}{l_{\nu+n}^{p'}} \right]^{1/p'}$$

$$= O\left[l_m^{\alpha+1+1/p} \frac{(l_m - l_n)^{1/p'}}{l_{n+1}} \right] \quad (n \to \infty)$$

$$= O\left[l_n^{\alpha+1+1/p} \frac{\{\varepsilon\, \theta(l_n)\}^{1/p'}}{l_n} \right] \tag{3.9}$$

where we have used the hypothesis (3.7) in the passage to the step preceding (3.9). Taking $m = n+1$ in the step preceding (3.9), we get also

$$a_{n+1} = O_R\left[l_n^{\alpha+1+1/p} \frac{(l_{n+1} - l_n)^{1/p'}}{l_{n+1}} \right] \quad (n \to \infty)$$

$$= O_R\left[l_{n+1}^{\alpha+1/p}\, l_{n+1}^{(r-\alpha-p^{-1}+\eta)/(r+1-p^{-1})p'} \right]$$

$$= o_R\left[l_{n+1}^{\alpha+1/p} \{\theta(l_{n+1})\}^{1/p'} \right]. \tag{3.10}$$

From (3.9) and (3.10) with $n+1$ changed to n, we obtain, instead of (3.6) in the proof of Theorem III,

$$\overline{\lim_{n\to\infty}} \max_{l_n \leqq l_m < l_n + \varepsilon\theta(l_n)} \frac{a_n + a_{n+1} + \dots + a_m}{l_n^{\rho'}} = o_R(1), \quad \varepsilon \to 0,$$

where

$$\rho' = \alpha + \frac{1}{p} + \frac{(r' - \alpha - p^{-1} + \gamma)}{(r' + 1 - p^{-1})\, p'}.$$

After this the proof is completed exactly like that of Theorem III subsequent to (3.6).

It may be observed that the assumption $\alpha+1+p^{-1}\geqslant 0$ involves no loss of generality since $\alpha+1+p^{-1}<0$ makes successively $a_n+|a_n|\equiv 0$, $a_n\equiv 0$ and so $\sigma_r = -\infty$ for all $r\geqslant 0$.

THEOREM V. *In Theorem* II, *let hypothesis* (i) *be omitted on account of its being implicit* (*with* $q=0$, $\rho=\alpha+1$) *in hypothesis* (iii) *modified as under. Let hypothesis* (ii) *be retained with* ρ *changed to* $\alpha+1$, *and hypothesis* (iii) *replaced by*

$$a_n = O\left[l_n^{\alpha}(l_n-l_{n-1})\right]. \tag{3.11}$$

Then the conclusion is that $\Sigma a_n l_n^{-s}$ *is summable* (R, l_n, k), $0\leqslant k<r$, *for* σ *satisfying* (3.2).

THEOREM VI. *If, in Theorem* V, (3.11) *alone is changed to*

$$\sum_{\nu=1}^{n} |a_\nu|^p l_\nu^p (l_\nu - l_{\nu-1})^{1-p} = O\left[l_n^{p(\alpha+1)+1}\right],\ p>1,\ \alpha+1+p^{-1}\geqslant 0,$$

the conclusion will become the assertion that $\Sigma a_n l_n^{-s}$ *is summable* (R, l_n, k), $0\leqslant k<r$, *for* σ *satisfying* (3.8).

The proofs of Theorems V, VI are omitted, being obvious simplifications of those of Theorems III, IV, involving the use of Theorem I (A) with hypothesis (2.2) (b) instead of (2.2.) (a) as formerly. Theorems V and VI, as pointed out by Chandrasekharan and Minakshisundaram, yield Ananda Rau's and Ganapathy Iyer's extensions of the Schnee-Landau theorem when $\alpha\to+0$.

§ 4. FURTHER APPLICATIONS

Theorem I (A) is a base which, combined with Theorem B, produces Theorem II, and in this sense Theorem I (A) may be said to correspond to Theorem II. There are results corresponding to each of Theorems III-VI in the same sense. For instance, Deduction 1 below corresponds to Theorem III and shows how other deductions corresponding to Theorems IV-VI may be formulated. Deductions 2,3 are further examples of results based on Theorem I.

DEDUCTION 1. (A) *In Theorem* I (A), *suppose that* $\sigma_r<\alpha+1$ *and that* (2.2) (a) *is replaced by*

$$a_n = O_R\left[l_n^{\alpha}(l_n-l_{n-1})\right],\ l_n - l_{n-1} = O\left(l_n^{(r-\alpha+\sigma_r)/(r+1)}\right). \tag{4.1}$$