

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 15 (1969)
Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SOME CONVERSE THEOREMS ON THE ABSCISSAE OF SUMMABILITY OF GENERAL DIRICHLET SERIES
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Kapitel: §3. Applications to theorems of the Schnee-Landau type
DOI: <https://doi.org/10.5169/seals-43225>

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exactly as (2.2) (a) implies (2.7). Since now $\gamma > \sigma_r \geq \rho$, the above condition in its turn implies

$$\overline{\lim}_{n \rightarrow \infty} \max_{l_n \leq l_m < l_n + \varepsilon l_n} (b_n + b_{n+1} + \dots + b_m) = o_R(1), \quad \varepsilon \rightarrow 0.$$

By Theorem A with hypothesis (1.2) (a) and $a = b = 0$, it follows that $\Sigma a_n l_n^{-s}$ is convergent for any σ such that $\sigma \geq \gamma > \sigma_r$ and therefore $\sigma_0 \leq \sigma_r$. But, in any case, $\sigma_0 \geq \sigma_k \geq \sigma_r$ for $0 \leq k < r$ and so we have the conclusion (2.5).

In the preceding argument we have supposed that $\sigma_r < \infty$ since $\sigma_r = \infty$ implies trivially $\sigma_k = \infty$.

§ 3. APPLICATIONS TO THEOREMS OF THE SCHNEE-LANDAU TYPE

Theorem II given next is the simplest of the theorems of the type mentioned above and it is a direct combination of Theorems I, B. Theorems V, VI are generalizations, respectively of Ananda-Rau's and Ganapathy Iyer's extensions of the Schnee-Landau theorem ([2], Theorem 9; [7], Theorem 10), as given by Chandrasekharan and Minakshisundaram ([6], pp. 88-9, Corollaries 3.73, 3.74). Theorems III, IV are apparently new counterparts of Theorems V, VI, the newness consisting in the replacement of the two-sided Tauberian conditions of the latter pair of theorems by analogous one-sided conditions suitably supplemented.

THEOREM II. Suppose that (i) the Dirichlet series,

$$\sum_1^{\infty} \frac{a_n}{l_n^s}, \quad s = \sigma + i\tau,$$

is summable (R, l_n, q) for some $q \geq 0$ when $\sigma > \rho$, (ii) the sum-function $f(s)$ thus defined is regular for $\sigma > \eta$ when $\eta < \rho$, and satisfies the condition

$$f(s) = O(|\tau|^r), \quad r > 0, \quad \text{uniformly for } \sigma \geq \eta + \varepsilon > \eta,$$

(iii) the coefficients a_n of the Dirichlet series satisfy ONE of the two alternatives (a), (b) of (2.2), but with $\theta(x) \equiv x^{1-(\rho-\eta)/r}$. Then the Dirichlet series is summable (R, l_n, k) , $0 \leq k < r$, for

$$\sigma \geq \frac{(r-k)\rho + k\eta}{r}.$$

Proof. By Theorem B, the Dirichlet series is summable (R, l_n, r') , $r' > r$, for $\sigma > \eta$ and hence $\sigma_{r'} \leq \eta < \rho$. Therefore it is evident from the proof of

Theorem I (A) ending with (2.10) that the Dirichlet series is summable (R, l_n, k) , $0 \leq k < r'$, for

$$\sigma \geq \frac{(r' - k) \rho + k \eta}{r'},$$

whence the desired conclusion follows when we let $r' \rightarrow r$.

THEOREM III. In Theorem II, let ρ be replaced by $\alpha + 1$ in hypotheses (i) and (ii); also let hypothesis (iii) be replaced by

$$a_n = O_R[l_n^\alpha (l_n - l_{n-1})], \quad l_n - l_{n-1} = O\left(l_n^{\frac{r - \alpha + \eta}{r + 1}}\right). \quad (3.1)$$

Then the conclusion is that $\Sigma a_n l_n^{-s}$, $s = \sigma + i\tau$, is summable (R, l_n, k) , $0 \leq k < r$, for

$$\sigma > \frac{(r - k)(\alpha + 1) + (k + 1)\eta}{r + 1}. \quad (3.2)$$

Proof. As in the proof of Theorem II, the series $\Sigma a_n l_n^{-s}$ is summable (R, l_n, r') , $r' > r$, for $\sigma > \eta$ where now $\eta < \alpha + 1$, so that $\sigma_{r'} \leq \eta < \alpha + 1$. We begin by choosing γ and correspondingly $\theta(x)$ as follows:

$$\eta < \gamma < \alpha + 1, \quad \theta(x) \equiv x^{(r' - \alpha + \gamma)/(r' + 1)}. \quad (3.3)$$

Then, since $r' > r$ and $\gamma > \eta$, we have

$$\frac{r' - \alpha + \gamma}{r' + 1} > \frac{r - \alpha + \gamma}{r + 1} > \frac{r - \alpha + \eta}{r + 1}.$$

And so (3.1) gives us, as $n \rightarrow \infty$,

$$a_n = O_R\left[l_n^\alpha l_n^{\frac{r - \alpha + \eta}{r + 1}}\right] = O_R\left[l_n^\alpha l_n^{\frac{r' - \alpha + \gamma}{r' + 1}}\right] = O_R[l_n^\alpha \theta(l_n)]. \quad (3.4)$$

Also, if $l_n \leq l_m < l_n + \varepsilon \theta(l_n)$, (3.1) again gives us as $n \rightarrow \infty$,

$$a_{n+1} + a_{n+2} + \dots + a_m = \begin{cases} O_R[l_m^\alpha (l_m - l_n)] & \text{if } \alpha \geq 0, \\ O_R[l_n^\alpha (l_m - l_n)] & \text{if } \alpha < 0, \end{cases}$$

so that, whether $\alpha \geq 0$ or $\alpha < 0$,

$$a_{n+1} + a_{n+2} + \dots + a_m = O_R[l_n^\alpha \varepsilon \theta(l_n)]. \quad (3.5)$$

In (3.4) and (3.5),

$$l_n^\alpha \theta(l_n) = l_n^{\rho'} \quad \text{where} \quad \rho' = \alpha + \frac{r' - \alpha + \gamma}{r' + 1} (> \gamma).$$

Hence, combining (3.4) and (3.5), we get

$$\lim_{n \rightarrow \infty} \max_{l_n \leq l_m < l_n + \varepsilon \theta(l_n)} \frac{a_n + a_{n+1} + \dots + a_m}{l_n^{\rho'}} = o_R(1), \quad \varepsilon \rightarrow 0. \quad (3.6)$$

(3.6) and the fact, following from Theorem B, that $\Sigma a_n l_n^{-s}$ is summable (R, l_n, r') , enables us to use (2.10) in the proof of Theorem I (A) with r, ρ replaced by r', ρ' respectively, so as to infer that $\Sigma a_n l_n^{-s}$ is summable (R, l_n, k) , $0 \leq k < r'$, for

$$\sigma \geq \frac{(r' - k) \rho' + k\gamma}{r'} = \frac{(r' - k)(\alpha + 1) + (k + 1)\gamma}{r' + 1}.$$

This yields (3.2) as required when we let $r' \rightarrow r$ and recall that $\gamma (> \eta)$ can be taken arbitrarily close to η .

THEOREM IV. *In Theorem III, (3.1) alone can be changed to*

$$\left. \begin{aligned} \sum_{v=1}^n (a_v + |a_v|) l_v^p (l_v - l_{v-1})^{1-p} &= O(l_n^{p(\alpha+1)+1}), \quad l_n - l_{n-1} = \\ &= O\left[l_n^{\frac{r-\alpha-p^{-1}+\eta}{r+1-p^{-1}}}\right], \quad p > 1, \quad \alpha+1 + p^{-1} \geq 0, \end{aligned} \right\} \quad (3.7)$$

with the conclusion changed in consequence to the assertion that $\Sigma a_n l_n^{-s}$ is summable (R, l_n, k) , $0 \leq k < r$, for

$$\sigma > \frac{(r-k)(\alpha+1) + (k+1-p^{-1})\eta}{r+1-p^{-1}}. \quad (3.8)$$

Proof. We observe that Theorem III may be viewed as the limiting case $p = \infty$ of Theorem IV.

The proof itself is similar to that of Theorem III with the difference that the choice of γ and $\theta(x)$ in (3.3) is now altered as below:

$$\eta < \gamma < \alpha + 1, \quad \theta(x) \equiv x^{(r'-\alpha-p^{-1}+\gamma)/(r'+1-p^{-1})}$$

¹⁾ We suppose that $l_0 = 0$.

And furthermore the step corresponding to (3.6) is obtained as follows. Writing $1 - 1/p = 1/p'$, we get, for $l_n \leq l_m < l_n + \varepsilon \theta(l_n)$,

$$\begin{aligned}
 a_{n+1} + a_{n+2} + \dots + a_m &\leq a_{n+1} + |a_{n+1}| + \dots + a_m + |a_m| \\
 &= \sum_{v=1}^{m-n} (a_{v+n} + |a_{v+n}|) l_{v+n} (l_{v+n} - l_{v+n-1})^{(1-p)/p} \times \\
 &\quad \times \frac{(l_{v+n} - l_{v+n-1})^{1/p'}}{l_{v+n}} \\
 &\leq \left[\sum_{v=1}^{m-n} (a_{v+n} + |a_{v+n}|)^p l_{v+n}^p (l_{v+n} - l_{v+n-1})^{1-p} \right]^{1/p} \times \\
 &\quad \times \left[\sum_{v=1}^{m-n} \frac{l_{v+n} - l_{v+n-1}}{l_{v+n}^{p'}} \right]^{1/p'} \\
 &= O \left[l_m^{\alpha+1+1/p} \frac{(l_m - l_n)^{1/p'}}{l_{n+1}} \right] (n \rightarrow \infty) \\
 &= O \left[l_n^{\alpha+1+1/p} \frac{\{\varepsilon \theta(l_n)\}^{1/p'}}{l_n} \right] \quad (3.9)
 \end{aligned}$$

where we have used the hypothesis (3.7) in the passage to the step preceding (3.9). Taking $m = n+1$ in the step preceding (3.9), we get also

$$\begin{aligned}
 a_{n+1} &= O_R \left[l_n^{\alpha+1+1/p} \frac{(l_{n+1} - l_n)^{1/p'}}{l_{n+1}} \right] (n \rightarrow \infty) \\
 &= O_R \left[l_{n+1}^{\alpha+1/p} l_{n+1}^{(r-\alpha-p^{-1}+\eta)/(r+1-p^{-1})p'} \right] \\
 &= o_R \left[l_{n+1}^{\alpha+1/p} \{\theta(l_{n+1})\}^{1/p'} \right]. \quad (3.10)
 \end{aligned}$$

From (3.9) and (3.10) with $n+1$ changed to n , we obtain, instead of (3.6) in the proof of Theorem III,

$$\overline{\lim}_{n \rightarrow \infty} \max_{l_n \leq l_m < l_n + \varepsilon \theta(l_n)} \frac{a_n + a_{n+1} + \dots + a_m}{l_n^{\rho'}} = o_R(1), \quad \varepsilon \rightarrow 0,$$

where

$$\rho' = \alpha + \frac{1}{p} + \frac{(r' - \alpha - p^{-1} + \gamma)}{(r' + 1 - p^{-1})p'}.$$

After this the proof is completed exactly like that of Theorem III subsequent to (3.6).

It may be observed that the assumption $\alpha+1+p^{-1} \geq 0$ involves no loss of generality since $\alpha+1+p^{-1} < 0$ makes successively $a_n + |a_n| \equiv 0$, $a_n \equiv 0$ and so $\sigma_r = -\infty$ for all $r \geq 0$.

THEOREM V. *In Theorem II, let hypothesis (i) be omitted on account of its being implicit (with $q = 0$, $\rho = \alpha+1$) in hypothesis (iii) modified as under. Let hypothesis (ii) be retained with ρ changed to $\alpha+1$, and hypothesis (iii) replaced by*

$$a_n = O[l_n^\alpha(l_n - l_{n-1})]. \quad (3.11)$$

Then the conclusion is that $\Sigma a_n l_n^{-s}$ is summable (R, l_n, k) , $0 \leq k < r$, for σ satisfying (3.2).

THEOREM VI. *If, in Theorem V, (3.11) alone is changed to*

$$\sum_{v=1}^n |a_v|^p l_v^p (l_v - l_{v-1})^{1-p} = O[l_n^{p(\alpha+1)+1}], \quad p > 1, \quad \alpha + 1 + p^{-1} \geq 0,$$

the conclusion will become the assertion that $\Sigma a_n l_n^{-s}$ is summable (R, l_n, k) , $0 \leq k < r$, for σ satisfying (3.8).

The proofs of Theorems V, VI are omitted, being obvious simplifications of those of Theorems III, IV, involving the use of Theorem I (A) with hypothesis (2.2) (b) instead of (2.2.) (a) as formerly. Theorems V and VI, as pointed out by Chandrasekharan and Minakshisundaram, yield Ananda Rau's and Ganapathy Iyer's extensions of the Schnee-Landau theorem when $\alpha \rightarrow +0$.

§ 4. FURTHER APPLICATIONS

Theorem I (A) is a base which, combined with Theorem B, produces Theorem II, and in this sense Theorem I (A) may be said to correspond to Theorem II. There are results corresponding to each of Theorems III-VI in the same sense. For instance, Deduction 1 below corresponds to Theorem III and shows how other deductions corresponding to Theorems IV-VI may be formulated. Deductions 2,3 are further examples of results based on Theorem I.

DEDUCTION 1. (A) *In Theorem I (A), suppose that $\sigma_r < \alpha+1$ and that (2.2) (a) is replaced by*

$$a_n = O_R[l_n^\alpha(l_n - l_{n-1})], \quad l_n - l_{n-1} = O(l_n^{(r-\alpha+\sigma_r)/(r+1)}). \quad (4.1)$$