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Artikel: SOME CONVERSE THEOREMS ON THE ABSCISSAE OF SUMMABILITY OF GENERAL DIRICHLET SERIES

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of general type λ and some *positive integer* order. These later versions are proved by him by using a difference formula applicable to such an integral mean ([9], Lemma 2); and each of them has a hypothesis which is an extension of the one-sided or two-sided Schmidt condition of slow growth of a function. Theorem A is a reformulation of Karamata's later theorems for any Riesz mean of a sequence, of general type λ and some *positive non-integer order*. In its fundamental case, $a = b = 0$, Theorem A has an analogue for the Abel mean of type λ instead of a Riesz mean of type λ, consisting of a classical theorem ([5], Theorem E) and Bosanquet's addition thereto ([5], Theorem D). Theorem A itself has been proved by me ([12], Theorem VI) by means of certain difference formulae due to Bosanquet ([4], Theorem 1) which extend Karamata's difference formula just mentioned to an integral mean of *non-integer* order. Bosanquet first proved his extended difference formulae in 1943, independently of Karamata. But, as a matter of fact, he had used them much earlier in 1931 in a form equivalent to Karamata's ([3], Lemma 5). To complete the references in relation to Bosanquet's difference formulae, mention may be made of certain other difference formulae independently evolved by Minakshisundaram and myself ([10], formulae (2.32), (2.38)) which are serviceable for much the same purposes as Bosanquet's formulae.

This paper deals specifically with general Dirichlet series of type l as distinguished from those of type λ. However, as far as Riesz typical means alone are concerned, there is no distinction between means of the two types, and so (for convenience) the Riesz means of this paper are taken to be of type l or (more explicitly) of type l_n, where l or l_n $(n = 1, 2, \ldots)$ is a divergent sequence strictly increasing and positive.

§ 1. Notation and auxiliary results

Let $a_1 + a_2 + \ldots$ be a real series and l a sequence $\{l_n\}$ such that

$$1 \leqslant l_1 < l_2 < \ldots, \quad l_n \to \infty .$$

Then, as usual, we define the Riesz mean of Σa_n of type l or l_n and order $r > 0$ by

$$\int_0^x \left(1 - \frac{t}{x}\right)^r dA_l(t) = \frac{r}{x^r} \int_0^x (x-t)^{r-1} A_l(t)\, dt \equiv \frac{A_l^r(x)}{x^r},$$

where $A_l^r(x)$ is the usual Riesz sum of Σa_n of type l or l_n and order r,

$$A_l(t) = a_1 + a_2 + \ldots + a_n \quad \text{for} \quad l_n \leqslant t < l_{n+1} \ (n \geqslant 1),$$

$$A_l(t) = 0 \quad \text{for} \quad t < l_1 .$$

Again, as usual, we define $A_l^0(t) = A_l(t)$ and define as follows summability of Σa_n to sum S by the Riesz mean of type l_n and order $r \geqslant 0$, briefly called summability (R, l_n, r) of Σa_n to S:

$$\frac{A_l^r(x)}{x^r} \to S, \text{ or } A_l^r(x) - Sx^r = o(x^r), \ x \to \infty .$$

In using this definition we may suppose (without loss of generality) that $S = 0$ since this merely means our considering $\Sigma a_n - S$ instead of Σa_n. Furthermore, when considering any other series $b_1 + b_2 + \ldots$, it is convenient to denote by $B_l^r(x)$, $r \geqslant 0$, the Riesz sum for that series, defined exactly as $A_l^r(x)$ for Σa_n.

In the usual notation again, the general Dirichlet series of type l or l_n, with coefficients $\{a_n\}$, is

$$\sum_1^\infty \frac{a_n}{l_n^s}, \quad s = \sigma + i\tau .$$

Corresponding to the summability (R, l_n, r), $r \geqslant 0$, of this series, to sum-function $f(s)$, we have the abscissa of summability σ_r $(-\infty < \sigma_r < \infty)$ characterized by the property that the series is summable (R, l_n, r), or not summable (R, l_n, r), according as $\sigma > \sigma_r$ or $\sigma < \sigma_r$.

In the above notation, we may state as under the lemmas and auxiliary theorems used in this paper, denoting Riesz sums of order $r \geqslant 0$, of Σa_n and Σb_n respectively, by $A^r(x)$ and $B^r(x)$, with omission of the suffix l indicative of the type which remains the same throughout.

LEMMA 1 ([1], Theorem 6; [2], Theorem 1). *Let* $\Sigma b_n \equiv \Sigma a_n l_n^\gamma$, *where* $\gamma > 0$ *is a constant. If* $A^r(x) = o(x^\beta)$, $x \to \infty$, *where* $\beta \geqslant r \geqslant 0$, *then* $B^r(x) = o(x^{\beta+\gamma})$.

LEMMA 2 ([1], Theorem 9; [2], Theorem 3). *If* $A^k(x) = o(x^{k+\beta})$, $x \to \infty$, *where* $k \geqslant 0$, $\beta > 0$, *then* $\Sigma b_n \equiv \Sigma a_n l_n^{-\beta}$ *is either summable* (R, l_n, k) *or never summable* (R, l_n, r) *for any r however large.*

LEMMA 3 ([1], Theorem 4; [2], Theorem II). *If*

$$A^r(x) = o\{W(x)\} \, (r > 0), \ A(x) = O\{V(x)\}, \ x \to \infty ,$$

where $W(x)$, $V(x)$ *are positive monotonic increasing functions of* $x>0$, *then, for* $0<k<r$, $A^k(x) = o(V^{1-k/r}\, W^{k/r})$ *where* $V = V(x)$, $W = W(x), x \to \infty$.

Lemma 3 is proved, in the papers referred to ([1], [2]), for any integrable function $\phi(x)$ instead of $A(x) \equiv \sum\limits_{l_n \leqq x} a_n$.

THEOREM A ([12], Theorem VI). *If*

$$A^r(x) = o(x^{r+b}), \ x \to \infty, \ \textit{where}\ r > 0, \ r+b \geqslant 0, \tag{1.1}$$

and if, with

$$\theta(x) \equiv x^{1-(a-b)/r}, \ a \geqslant b,$$

we have

$$\left.\begin{array}{l}
\text{EITHER (a)} \\[4pt]
\overline{\lim\limits_{n\to\infty}} \max\limits_{l_n \leqq l_m < l_n + \varepsilon\theta(l_n)} \dfrac{a_n + a_{n+1} + \ldots + a_m}{l_n^a} = o_R(1), \ \varepsilon \to 0, \\[12pt]
\text{OR (b)} \\[4pt]
\overline{\lim\limits_{n\to\infty}} \max\limits_{l_n \leqq l_m < l_n + \varepsilon\theta(l_n)} \dfrac{|a_{n+1} + a_{n+2} + \ldots + a_m|}{l_n^a} = o(1), \ \varepsilon \to 0, {}^{1)}
\end{array}\right\} \tag{1.2}$$

then

$$A(l_n) = o(l_n^a), \ n \to \infty.$$

THEOREM B (Riesz; see e.g. [6], p. 81, Theorem 3.66). *Suppose that the Dirichlet series*

$$\sum_1^\infty \frac{a_n}{l_n^s}, \quad s = \sigma + i\tau,$$

is summable (R, l_n, q) *for some* $q \geqslant 0$ *when* $\sigma > d$. [2]) *Suppose also that the sum-function* $f(s)$ *thus defined is regular for* $\sigma > \eta$ *where* $\eta < d$, *and*

$$f(s) = O(|\tau|^r), \ r \geqslant 0, \quad \textit{uniformly for} \quad \sigma \geqslant \eta + \varepsilon > \eta.$$

Then the Dirichlet series is summable (R, l_n, r'), $r' > r$, *for* $\sigma > \eta$.

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1) $a_{n+1} + a_{n+2} + \ldots + a_m$ is to be interpreted as 0 when $n = m$ or $l_n = l_m$.
2) This is no restriction since otherwise $\sigma_q = \infty$ for all q.