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SOME CONVERSE THEOREMS ON THE ABSCISSAE OF SUMMABILITY OF GENERAL DIRICHLET SERIES

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To the memory of J. Karamata

Introduction

Chandrasekharan and Minakshisundaram have generalized ([6], p. 21, Theorem 1.82) a fundamental theorem which asserts the convergence of a series when the series is (i) summable by a Riesz mean of general type λ and some positive order, (ii) subject to an appropriate Tauberian condition in two-sided Schmidt form. Basing themselves on their generalization, they have extended at one stroke ([6], pp. 86, 88, Theorems 3.71, 3.72), certain converse theorems on the abscissae of summability of general Dirichlet series, due in the first instance to Ananda-Rau ([2], Theorems 7, 8, 9) with Tauberian conditions on individual coefficients of the series, and due subsequently to Ganapathy Iyer ([7], Theorems 7, 8, 10) with Tauberian conditions formally including those of Ananda-Rau. Now the fundamental theorem generalized by Chandrasekharan and Minakshisundaram contains, besides the two-sided Schmidt hypothesis taken into account by them, an alternative one-sided hypothesis. And this theorem in its entirety, with both alternative hypotheses, has a natural generalization in Theorem A (§ 1) of which it is, in fact, the special case $a = b = 0$. In the present context the significance of Theorem A lies in its being a basis, not only for the extensions of Ananda-Rau's and Ganapathy Iyer's theorems given by Chandrasekharan and Minakshisundaram, but also for some further extensions of the same type (§§ 2, 3, 4).

It is relevant to mention here that the earliest version of Theorem A is due to Karamata ([8], § 1.1) and concerned with the Cesàro first-order mean of a series or sequence in place of a Riesz mean of general type λ and some positive order. Two later versions, also due to Karamata and found in a paper by him dated November 1939 ([9], Théorèmes 1a), 3f)), are concerned with an integral mean including as a special case a Riesz mean of general type λ and some *positive integer* order. These later versions are proved by him by using a difference formula applicable to such an integral mean ([9], Lemma 2); and each of them has a hypothesis which is an extension of the one-sided or two-sided Schmidt condition of slow growth of a function. Theorem A is a reformulation of Karamata's later theorems for any Riesz mean of a sequence, of general type λ and some *positive non-integer order*. In its fundamental case, $a = b = 0$, Theorem A has an analogue for the Abel mean of type λ instead of a Riesz mean of type λ, consisting of a classical theorem ([5], Theorem E) and Bosanquet's addition thereto ([5], Theorem D). Theorem A itself has been proved by me ([12], Theorem VI) by means of certain difference formulae due to Bosanquet ([4], Theorem 1) which extend Karamata's difference formula just mentioned to an integral mean of *non-integer* order. Bosanquet first proved his extended difference formulae in 1943, independently of Karamata. But, as a matter of fact, he had used them much earlier in 1931 in a form equivalent to Karamata's ([3], Lemma 5). To complete the references in relation to Bosanquet's difference formulae, mention may be made of certain other difference formulae independently evolved by Minakshisundaram and myself ([10], formulae (2.32), (2.38)) which are serviceable for much the same purposes as Bosanquet's formulae.

This paper deals specifically with general Dirichlet series of type l as distinguished from those of type λ. However, as far as Riesz typical means alone are concerned, there is no distinction between means of the two types, and so (for convenience) the Riesz means of this paper are taken to be of type l or (more explicitly) of type l_n, where l or l_n $(n = 1, 2, ...)$ is a divergent sequence strictly increasing and positive.

§ 1. Notation and auxiliary results

Let $a_1 + a_2 + ...$ be a real series and l a sequence $\{l_n\}$ such that

$$1 \leqslant l_1 < l_2 < \dots, \quad l_n \to \infty .$$

Then, as usual, we define the Riesz mean of Σa_n of type l or l_n and order $r > 0$ by

$$\int_0^x \left(1 - \frac{t}{x}\right)^r dA_l(t) = \frac{r}{x^r} \int_0^x (x-t)^{r-1} A_l(t)\, dt \equiv \frac{A_l^r(x)}{x^r},$$

where $A_l^r(x)$ is the usual Riesz sum of Σa_n of type l or l_n and order r,