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# ASYMPTOTIC BEHAVIOUR OF A CLASS OF DETERMINANTS

by Mark KAC

*To the memory of J. Karamata*

1° Let  $f(\theta)$ ,  $0 < \theta < 1$ , be a function which in addition to certain regularity conditions, which will be specified later on, is bounded from below by a number larger than 2. In other words,

$$(1.1) \quad f(\theta) > m > 2, \quad 0 < \theta < 1.$$

Consider the determinant

$$(1.2) \quad D_n(f) = \begin{vmatrix} f\left(\frac{1}{n}\right), & -1, & 0, & \dots & - & \dots & 0 \\ -1, & f\left(\frac{2}{n}\right), & -1, & \dots & & & 0 \\ 0, & -1, & f\left(\frac{3}{n}\right), & -1, & \dots & & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & \dots & & -1, & f\left(\frac{n}{n}\right) & \dots \end{vmatrix}$$

i.e. the determinant of the matrix

$$(1.3) \quad a_{ij} = f\left(\frac{i}{n}\right) \delta_{ij} - \delta_{i,j-1} - \delta_{i,j+1}$$

where  $\delta$ , as usual, denotes the Kronecker delta.

If  $f(\theta)$  is Riemann integrable it can be shown that

$$(1.4) \quad \lim_{n \rightarrow \infty} D_n^n(f) = \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} d\omega \int_0^1 \log(f(\theta) - 2 \cos \omega) d\theta \right\} = G(f).$$

This is a special case of a more general result of Kac, Murdock and Szegô [1953].

Assuming that  $f(\theta)$  is twice differentiable, with a bounded second derivative, one can go a good deal farther.

In fact, one can determine

$$(1.5) \quad \lim_{n \rightarrow \infty} \frac{D_n(f)}{G^n(f)}$$

with  $G(f)$  as defined in (1.4).

Although the result is again a special case of a more general result obtained by Mejlbo and Schmidt [1962] the method which we shall use is quite different and of some interest in itself. The method has also the advantage of explaining the role of smoothness conditions imposed on  $f$ .

2° We begin with the elementary formula

$$(2.1) \quad D_n^{-\frac{1}{2}}(f) = \left(\frac{1}{(\sqrt{\pi})^n}\right)^n \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp \left\{ - \sum_1^n f\left(\frac{k}{n}\right) x_k^2 + 2 \sum_1^{n-1} x_k x_{k+1} \right\} dx_1 dx_2 \dots dx_n$$

and note that setting

$$(2.2) \quad K\left(x, y; \frac{l}{n}\right) = \frac{1}{\sqrt{\pi}} \exp \left\{ - \frac{1}{2} f\left(\frac{l}{n}\right) x^2 + 2xy - \frac{1}{2} f\left(\frac{l+1}{n}\right) y^2 \right\}$$

we can rewrite (2.1) in the form

$$(2.3) \quad D_n^{-\frac{1}{2}}(f) = \frac{1}{(\sqrt{\pi})^n} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp \left\{ - \frac{1}{2} f\left(\frac{1}{n}\right) x_1^2 \right\} K\left(x_1, x_2; \frac{1}{n}\right) K\left(x_2, x_3; \frac{2}{n}\right) \dots \dots K\left(x_{n-1}, x_n; \frac{n}{n}\right) \exp \left\{ - \frac{1}{2} f\left(\frac{n}{n}\right) x_n^2 \right\} dx_1 dx_2 \dots dx_n.$$

The kernels  $K(x, y; \frac{l}{n})$  are not symmetric and we must replace them by symmetric ones without introducing too large an error.

This is done by noting that

$$(2.4) \quad f\left(\frac{l}{n}\right) = \frac{1}{2} f\left(\frac{l-1}{n}\right) + \frac{1}{4n} f'\left(\frac{l-1}{n}\right) + \frac{1}{2} f\left(\frac{l}{n}\right) + \frac{1}{4n} f'\left(\frac{l}{n}\right) + 0\left(\frac{1}{n^2}\right)$$

where the error  $0(1/n^2)$  is actually less than  $M/n^2$  where  $M$  is a bound for  $f''(\theta)$  in  $(0, 1)$ .

It follows that

$$(2.5) \quad \left| \sum_{l=1}^n f\left(\frac{l}{n}\right) x_l^2 - \left\{ \left( \frac{1}{2} f(0) + \frac{1}{4n} f'(0) \right) x_1^2 + \right. \right. \\ \left. \left. + \sum_{l=1}^{n-1} \left[ \left( \frac{1}{2} f\left(\frac{l}{n}\right) + \frac{1}{4n} f'\left(\frac{l}{n}\right) \right) x_l^2 + \left( \frac{1}{2} f\left(\frac{l}{n}\right) + \frac{1}{4n} f'\left(\frac{l}{n}\right) \right) x_{l+1}^2 \right] \right. \right. \\ \left. \left. + \left( \frac{1}{2} f(1) + \frac{1}{4n} f'(1) \right) x_n^2 \right\} \right| < \frac{M}{n^2} \sum_1^n x_l^2$$

and this suggests that we introduce the symmetric kernels

$$(2.6) \quad \tilde{K}_\alpha \left( x, y; \frac{l}{n} \right) = \frac{1}{\sqrt{\pi}} \exp \left\{ - \left( \frac{1}{2} f\left(\frac{l}{n}\right) + \frac{1}{4n} f'\left(\frac{l}{n}\right) + \frac{\alpha}{2n^2} \right) x^2 + 2xy \right. \\ \left. - \left( \frac{1}{2} f\left(\frac{l}{n}\right) + \frac{1}{4n} f'\left(\frac{l}{n}\right) + \frac{\alpha}{2n^2} \right) y^2 \right\}.$$

It should now be clear from (2.3) and (2.5) that  $D_n^{-\frac{1}{2}}(f)$  is contained between the value of the integral

$$(2.7) \quad \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp \left\{ - \left( \frac{1}{2} f(0) + \frac{1}{4n} f'(0) \right) x_1^2 \right\} \prod_{l=1}^{n-1} K_\alpha \left( x_l, x_{l+1}; \frac{l}{n} \right) \\ \exp \left\{ - \left( \frac{1}{2} f(1) + \frac{1}{4n} f'(1) \right) x_n^2 \right\} dx_1 \dots dx_n$$

for  $\alpha = -M$  and its value for  $\alpha = M$ .

3° It will turn out that the asymptotic behaviour of (2.7), to the required accuracy, does not depend on  $\alpha$  so that we may as well set  $\alpha = 0$ .

The kernels  $K_0(x, y; l/n)$  do not commute. However they “almost commute” in the following sense:

Let  $\lambda_1(l/n), \lambda_2(l/n), \dots$  be the eigenvalues of  $K_0(x, y; l/n)$  in the decreasing order and let

$$\varphi_1(x; l/n), \quad \varphi_2(x; l/n), \dots$$

be the corresponding normalized eigenfunctions.

Then

$$(3.1) \quad \int_{-\infty}^{\infty} \varphi_i \left( x; \frac{l}{n} \right) \varphi_j \left( x; \frac{l+1}{n} \right) dx = \delta_{ij} + o\left(\frac{1}{n^2}\right).$$

This follows by a perturbation argument but can also be obtained directly since the eigenfunctions and eigenvalues of our kernels can be determined explicitly.

I shall now ask the reader to believe me that whenever we have (3.1) and whenever  $\lambda_1(l/n) - \lambda_2(l/n)$  is bounded away from 0 (uniform non-degeneracy) the integral (2.7) divided by

$$(3.2) \quad \prod_{l=1}^{n-1} \lambda_1 \left( \frac{l}{n} \right)$$

approaches

$$(3.3) \quad \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} \varphi_1(x; 0) \exp \left\{ -\frac{1}{2} f(0) x^2 \right\} dx \int_{-\infty}^{\infty} \varphi_1(x; 1) \exp \left\{ -\frac{1}{2} f(1) x^2 \right\} dx .$$

In other words, the asymptotic behaviour of the integral (2.7) is the same as if the kernels  $K_0(x, y; l/n)$  commuted. The details of the proof are tedious but intuitively the assertions should be nearly obvious.

It remains now to calculate  $\lambda_1(l/n)$  and  $\varphi_1(x; l/n)$ .

The kernels  $K_0$  are of the form

$$(3.4) \quad \frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{a}{2} x^2 + 2xy - \frac{a}{2} y^2 \right\}$$

with  $a > 2$  and it is easy to convince oneself that the principal eigenfunction is of the form

$$(3.5) \quad \frac{\sqrt[4]{b}}{\sqrt[4]{\pi}} \exp \left\{ -\frac{b}{2} y^2 \right\}$$

A simple calculation then gives

$$(3.6) \quad b = \sqrt{a^2 - 4}$$

and for the maximum eigenvalue  $\lambda_1(a)$  the formula

$$(3.7) \quad \lambda_1(a) = \left( \frac{a}{2} + \sqrt{\left( \frac{a}{2} \right)^2 - 1} \right)^{-\frac{1}{2}} .$$

Setting

$$(3.8) \quad a_l = f \left( \frac{l}{n} \right) + \frac{1}{2n} f' \left( \frac{l}{n} \right)$$

we finally obtain that

$$(3.9) \quad \lim_{n \rightarrow \infty} \frac{D_n(f)}{\prod_{l=1}^{n-1} \left( \frac{a_l}{2} + \sqrt{\left(\frac{a_l}{2}\right)^2 - 1} \right)} = \frac{f(0) + \sqrt{f^2(0) - 4}}{2 \sqrt[4]{f^2(0) - 4}} \cdot \frac{f(1) + \sqrt{f^2(1) - 4}}{2 \sqrt[4]{f^2(1) - 4}}.$$

It remains to determine the asymptotic behavior of the product

$$(3.10) \quad \prod_{l=1}^{n-1} \left( \frac{a_l}{2} + \sqrt{\left(\frac{a_l}{2}\right)^2 - 1} \right) = \exp \sum_{l=1}^{n-1} \log \left( \frac{a_l}{2} + \sqrt{\left(\frac{a_l}{2}\right)^2 - 1} \right).$$

Neglecting terms of order  $1/n^2$  and higher we find easily that

$$\log \left( \frac{a_l}{2} + \sqrt{\left(\frac{a_l}{2}\right)^2 - 1} \right) = \log \frac{f\left(\frac{l}{n}\right) + \sqrt{f^2\left(\frac{l}{n}\right) - 4}}{2} + \frac{1}{2n} \frac{f'\left(\frac{l}{n}\right)}{\sqrt{f^2\left(\frac{l}{n}\right) - 4}}$$

and hence

$$(3.11) \quad \sum_{l=1}^{n-1} \log \left( \frac{a_l}{2} + \sqrt{\left(\frac{a_l}{2}\right)^2 - 1} \right) = \sum_{l=1}^{n-1} \log \frac{f\left(\frac{l}{n}\right) + \sqrt{f^2\left(\frac{l}{n}\right) - 4}}{2} + \frac{1}{2} \int_0^1 \frac{f'(\theta)}{\sqrt{f^2(\theta) - 4}} d\theta + o\left(\frac{1}{n}\right).$$

Using the well known formula

$$(3.12) \quad \sum_{l=1}^{n-1} g\left(\frac{l}{n}\right) = n \int_0^1 g(\theta) d\theta - \frac{g(0) + g(1)}{2} + o\left(\frac{1}{n}\right)$$

for

$$(3.13) \quad g(\theta) = \log \frac{f(\theta) + \sqrt{f^2(\theta) - 4}}{2}$$

and noting that

$$\int_0^1 \frac{f'(\theta)}{\sqrt{f^2(\theta) - 4}} d\theta = \log \frac{f(1) + \sqrt{f^2(1) - 4}}{2} - \log \frac{f(0) + \sqrt{f^2(0) - 4}}{2}$$

we obtain

$$\sum_{l=1}^{n-1} \log \left( \frac{a_l}{2} + \sqrt{\left(\frac{a_l}{2}\right)^2 - 1} \right) =$$

$$(3.14) \quad n \int_0^1 \log \frac{f(\theta) + \sqrt{f^2(\theta) - 4}}{2} d\theta - \log \frac{f(0) + \sqrt{f^2(0) - 4}}{2} + o\left(\frac{1}{n}\right).$$

Combining (3.14) with (3.9) and noting that

$$\log G(f) = \frac{1}{2\pi} \int_0^1 d\theta \int_{-\pi}^{\pi} \log (f(\theta) - 2 \cos \omega) d\omega =$$

$$\int_0^1 \log \frac{f(\theta) + \sqrt{f^2(\theta) - 4}}{2} d\theta$$

we are led to the final result that

$$(3.15) \quad \lim_{n \rightarrow \infty} \frac{D_n(f)}{G^n(f)} = \frac{f(1) + \sqrt{f^2(1) - 4}}{\sqrt[4]{f^2(0) - 4} \sqrt[4]{f^2(1) - 4}}.$$

4° If we set

$$(4.1) \quad \log (f(\theta) - 2 \cos \omega) = \sum_{-\infty}^{\infty} h_\nu(\theta) e^{i\nu\omega}$$

the right hand side of (3.15) can be shown to be equal to

$$(4.2) \quad \exp \left\{ \frac{1}{2} (h_0(1) - h_0(0)) \right\} \exp \left\{ \frac{1}{2} \sum_1^{\infty} \nu h_\nu^2(0) + \frac{1}{2} \sum_1^{\infty} \nu h_\nu^2(1) \right\}.$$

This is in apparent disagreement with the formula of Mejlbo and Schmidt since in their formula  $h_0(1) - h_0(0)$  is replaced by  $h_0(0) - h_0(1)$ . We would get complete agreement if instead of our determinant (1.2) we would consider a modified determinant with  $f(0), f(\frac{1}{n}), f(\frac{2}{n}), \dots, f(\frac{n-1}{n})$  on the main diagonal. So accurate is formula (3.15) that it is sensitive to so slight a modification!

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