Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 15 (1969)
Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON ABSOLUTE SUMMABILITY FACTORS
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Kapitel: Notations
DOI: https://doi.org/10.5169/seals-43213

ON ABSOLUTE SUMMABILITY FACTORS

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To the memory of J. Karamata

In this paper we will give some results on absolute summability factors, i.e. on sequences $\{\varepsilon_n\}$ which transform—for given summability methods A and B—every absolutely A-summable series Σa_n into an absolutely B-summable series $\Sigma a_n \varepsilon_n$.

Theorems of this type are known for the Cesàro methods ([1], [2], [7], [8]), and we aim at proving corresponding theorems for matrix transforms in general. Some of the ideas which have been used in the Cesàro case can be employed in the general case and lead to Theorems 1 and 2 below (see also [3]). If we specialize in these theorems A to C_α and B to C_β $(0 \leqq \beta \leqq \alpha \leqq 1)$, then Theorem 1 covers the case $0 \leqq \beta < \alpha$, but not $\beta = \alpha$, and Theoiem 2 covers the case $0 < \beta \leqq \alpha$, but not $\beta = 0$. We give a third theorem which covers, when specialized as before, the case $0 \leqq \beta \leqq \alpha$, but the conditions imposed on A and B in this theorem appear to be more severe than in the other two theorems.

Each of the following theorems requires that the elements in the rows of A or B increase, and this leads for Cesàro methods to the restriction $\alpha \leqq 1$ or $\beta \leqq 1$. The question of an extension of the following theorems which contains the results for Cesàro methods of all orders remains open (see [3], [6]).

NOTATIONS

A matrix $A = (a_{n\nu})$ $(n, \nu = 0, 1, \ldots)$ is called triangular if $a_{n\nu} = 0$ for $\nu > n$, and normal, if it is triangular and if $a_{nn} \neq 0$. We write $A \stackrel{\cdot}{\geqq} 0$ $(A \stackrel{\cdot}{\leqq} 0)$ if $a_{n\nu} \geqq 0$ for $\nu \neq n$ $(a_{n\nu} \leqq 0$ for $\nu \neq n)$. The inverse of a normal matrix A is denoted by $A' = (a'_{n\nu})$, i.e. $AA' = I$, where I is the identity matrix. Given a matrix A and a series Σa_n, then Σa_n is called absolutely A-summable if

[1]) This work was supported in part by N.S.F. Grant GP 9026.

$\alpha_n = \sum_{\nu=0}^{\infty} a_{n\nu}\, a_\nu$ exists for $n = 0, 1, \ldots,$ and if $\Sigma\, |\alpha_n| < \infty$. We write $\Sigma a_n \in |A|$ if Σa_n is absolutely A-summable. A matrix A is called absolutely convergence preserving if $\Sigma a_n \in |A|$ whenever $\Sigma a_n \in |I|$, and absolutely regular, if in addition $\Sigma a_n = \Sigma \alpha_n$. Necessary and sufficient conditions that A preserves absolute convergence are (by a theorem of Knopp and Lorentz, [5])

$$\sum_{n=\mu}^{\infty} |a_{n\mu}| = O(1) \qquad (\mu \to \infty)\,, \tag{1}$$

and for absolute regularity we have to add the conditions

$$\sum_{n=\mu}^{\infty} a_{n\mu} = 1 \qquad (\mu = 0, 1, \ldots)\,. \tag{2}$$

If, for given matrices A and B, $\Sigma a_n \in |B|$ holds whenever $\Sigma a_n \in |A|$, then we write $|A| \subseteq |B|$.

Let A be normal, and let B be triangular. We write $\alpha_n = \sum_{\nu=0}^{n} a_{n\nu}\, a_\nu$, $\beta_n = \sum_{\nu=0}^{n} b_{n\nu}\, \varepsilon_\nu\, a_\nu = \sum_{\nu=0}^{n} b_{n\nu}\, \varepsilon_\nu \sum_{\mu=0}^{\nu} a'_{\nu\mu}\, a_\mu = \sum_{\mu=0}^{n} a_\mu \sum_{\nu=\mu}^{n} b_{n\nu}\, \alpha'_{\nu\mu}\, \varepsilon_\nu$ and observe that the sequence $\{\varepsilon_n)$ has the property that $\Sigma a_n\, \varepsilon_n \in |B|$ whenever $\Sigma a_n \in |A|$ if and only if the matrix $(\sum_{\nu=\mu}^{n} b_{n\nu}\, a'_{\nu\mu}\, \varepsilon_\nu)$ is absolute convergence preserving, i.e. if

$$\sum_{n=\mu}^{\infty} \Big| \sum_{\nu=\mu}^{n} b_{n\nu}\, a'_{\nu\mu}\, \varepsilon_\nu \Big| = O(1) \qquad (\mu \to \infty) \tag{3}$$

by (1). A sequence $\{\varepsilon_n\}$ is a sequence of absolute summability factors from A to B if and only if (3) is satisfied. The following theorem gives necessary conditions for factors of this type. The proof is very similar to the proof of Theorem 1 in [3] and is omitted.

THEOREM. *Let A be normal and absolutely convergence preserving, and let B be triangular and absolutely regular. Then, if $\Sigma a_n\, \varepsilon_n \in |B|$ whenever $\Sigma a_n \in |A|$, it follows that*

$$\varepsilon_\nu = \sum_{n=\nu}^{\infty} b_n\, a_{n\nu} \quad \textit{for some bounded sequence } \{b_n\}\,, \tag{4}$$

$$\frac{b_{nn}}{a_{nn}}\, \varepsilon_n = O(1)\,. \tag{5}$$

If A and B are Cesàro matrices, then conditions (4) and (5) are sufficient for (3), and in what follows we will show that this is also the case for more general matrices A and B. It will be convenient to use the notation $\varepsilon_\nu(A, b)$ for the infinite series which appears in (4).

THEOREM 1. *Let A be normal and absolutely regular, and let B be triangular and absolutely regular. Furthermore, assume that* $\sum_{\nu=\mu}^{\infty} |a'_{\nu\mu}| < \infty$ $(\mu = 0, 1, \dots)$ *and* $0 \leqq b_{n\nu} \uparrow$ *for* $\nu \uparrow$ $(\nu \leqq n)$. *Then* $\Sigma a_\nu \varepsilon_\nu(A, b) \in |B|$ *whenever* $\Sigma v_\nu \in |A|$ if [1])

$$(6) \qquad \sum_{\nu=\mu+1}^{\infty} |a'_{\nu\mu}\, \varepsilon_\nu(A, b)|\, \bar{b}_{\nu-1,\mu} = O(1) \quad (\mu \to \infty), \qquad \bar{b}_{n\mu} = \sum_{\nu=\mu}^{n} b_{\nu\mu}.$$

Proof. We have to show that (3) holds, and we write $\varepsilon_\nu = \varepsilon_\nu(A, b)$. Let (for $n \geqq \mu$)

$$\sum_{\nu=\mu}^{n} b_{n\nu}\, a'_{\nu\mu}\, \varepsilon_\nu = b_{n\mu} \sum_{\nu=\mu}^{\infty} a'_{\nu\mu}\, \varepsilon_\nu - b_{n\mu} \sum_{\nu=n+1}^{\infty} a'_{\nu\mu}\, \varepsilon_\nu + \sum_{\nu=\mu}^{n} (b_{n\nu} - b_{n\mu})\, a'_{\nu\mu}\, \varepsilon_\nu =$$

$$= I + II + III.$$

Here

$$\sum_{\nu=\mu}^{\infty} a'_{\nu\mu}\, \varepsilon_\nu = \sum_{\nu=\mu}^{\infty} a'_{\nu\mu} \sum_{\rho=\nu}^{\infty} b_\rho\, a_{\rho\nu} = \sum_{\rho=\mu}^{\infty} b_\rho \sum_{\nu=\mu}^{\rho} a_{\rho\nu}\, a'_{\nu\mu} = b_\mu,$$

and it follows that

$$\sum_{n=\mu}^{\infty} |I| = O(1) \quad (\mu \to \infty).$$

Next

$$\sum_{n=\mu}^{\infty} |III| \leqq \sum_{\nu=\mu}^{\infty} |a'_{\nu\mu}\, \varepsilon_\nu| \sum_{n=\nu}^{\infty} (b_{n\nu} - b_{n\mu}) = \sum_{\nu=\mu}^{\infty} |a'_{\nu\mu}\, \varepsilon_\nu| \sum_{n=\mu}^{\nu-1} b_{n\mu}$$

(observe (2)), and finally

$$\sum_{n=\mu}^{\infty} |II| \leqq \sum_{\nu=\mu+1}^{\infty} |a'_{\nu\mu}\, \varepsilon_\nu| \sum_{n=\mu}^{\nu-1} b_{n\mu},$$

which proves the theorem.

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[1]) If $\alpha_n = \sum_{\nu=0}^{n} b_{n\nu}\, a_\nu$, $\sigma_n = \alpha_0 + \dots + \alpha_n$, then $\sigma_n = \sum_{\nu=0}^{n} \bar{b}_{n\nu}\, a_\nu$, i.e. $(\bar{b}_{n\nu})$ is the form of the method defined by B which transforms a series into a sequence.

Remarks. Let $A = C_\alpha$ (i.e. $a_{n\nu} = \dfrac{A_{n-\nu}^{\alpha-1}\nu}{n A_n^\alpha}$, $1 \leqq \nu \leqq n$, $a_{no} = \delta_{no}$), $B = C_\beta$, then the assumptions of Theorem 1 are satisfied if $\alpha \geqq 0$, $0 \leqq \beta \leqq 1$. If $\beta \leqq \alpha$ and $\varepsilon_n(A, b) = O\left(\dfrac{a_{nn}}{b_{nn}}\right) = O(n^{\beta-\alpha})$, then (6) is the condition

$$(7)\qquad \mu A_\mu^\alpha \sum_{\nu=\mu+1}^{\infty} |A_{\nu-\mu}^{-\alpha-1} A_{\nu-\mu-1}^{\beta}|\, \nu^{\beta-\alpha} \frac{1}{\nu A_{\nu-1}^{\beta}} = O(1) \qquad (\mu \to \infty),$$

which is satisfied for $\beta < \alpha$ but not for $\beta = \alpha$.

THEOREM 2. *Let A be normal and absolutely regular, and let B be triangular and absolutely regular. Furthermore, assume that* [1] $A' \dot{\leqq} 0, a_n > 0, a_{n\nu}\uparrow$, $0 \leqq b_{n\nu}\uparrow$ for $\nu \uparrow (\nu \leqq n)$. *Then $\Sigma a_\nu \varepsilon_\nu(A, b) \in |B|$ whenever $\Sigma a_n \in |A|$ if*

$$(8)\qquad \varepsilon_\mu(A, b) \sum_{n=\mu}^{\infty} |(BA')_{n\mu}| = O(1) \qquad (\mu \to \infty)$$

and

$$(9)\qquad \sum_{n=\mu}^{\infty} |(BA')_{n\mu}|\, \bar{a}_{n\mu} = O(1) \qquad (\mu \to \infty), \qquad \bar{a}_{n\mu} = \sum_{\nu=\mu}^{n} a_{\nu\mu}.$$

Proof. We have to show that (3) holds, and we write $\varepsilon_\nu = \varepsilon_\nu(A, b)$. Let

$$\sum_{\nu=\mu}^{n} b_{n\nu} a'_{\nu\mu} \varepsilon_\nu = \varepsilon_\mu (BA')_{n\mu} + \sum_{\nu=\mu}^{n} a_{n\nu} a'_{\nu\mu}(\varepsilon_\nu - \varepsilon_\mu) = I + II.$$

Here $\sum_{n=\mu}^{\infty} |I| = O(1)$ because of (8).

We have

$$II = \sum_{\nu=\mu}^{n} b_{n\nu} a'_{\nu\mu}\Big(\sum_{\rho=\nu}^{\infty} b_\rho a_{\rho\nu} - \sum_{\rho=\mu}^{\infty} b_\rho a_{\rho\mu}\Big)$$

$$= \sum_{\rho=\mu}^{n} b_\rho \Big(\sum_{\nu=\mu}^{\rho} b_{n\nu} a_{\rho\nu} a'_{\nu\mu} - a_{\rho\mu}(BA')_{n\mu}\Big) + \sum_{\rho=n+1}^{\infty} b_\rho \sum_{\nu=\mu}^{n} b_{n\nu} a'_{\nu\mu}(a_{\rho\nu} - a_{\rho\mu}),$$

where

$$\sum_{\nu=\mu}^{\rho} b_{n\nu} a_{\rho\nu} a'_{\nu\mu} = \sum_{\nu=\mu}^{\rho} (b_{n\nu} - b_{n\mu}) a_{\rho\nu} a'_{\nu\mu} + b_{n\mu}\delta_{\rho\mu} \leqq 0$$

[1] It follows from $A' \dot{\leqq} 0$, $a_n > 0$ that $A \dot{\geqq} 0$ (see [4]).

for $\rho>\mu$, and $\sum_{\nu=\mu}^{n} b_{n\nu}\, a'_{\nu\mu}\,(a_{\rho\nu}-a_{\rho\mu})\leqq 0$. If $|b_n|\leqq K>0$, then it follows that

$$K^{-1}\,|II| \leqq -\sum_{\rho=\mu}^{n}\Big(\sum_{\nu=\mu}^{\rho} b_{n\nu}\, a_{\rho\nu}\, a'_{\nu\mu} - a_{\rho\mu}\,(BA')_{n\mu}\Big)$$

$$-\sum_{\rho=n+1}^{\infty}\sum_{\nu=\mu}^{n} b_{n\nu}\, a'_{\nu\mu}\,(a_{\rho\nu}-a_{\rho\mu}) + 2b_{n\mu} + 2\,|(BA')_{n\mu}|\sum_{\rho=\mu}^{n} a_{\rho\mu}$$

$$= -\sum_{\nu=\mu}^{n} b_{n\nu}\, a'_{\nu\mu}\,(\varepsilon_\nu(A,1)-\varepsilon_\mu(A,1)) + 2b_{n\mu} + 2\bar{a}_{n\mu}\,|(BA')_{n\mu}|\,,$$

and we have $\sum_{n=\mu}^{\infty}|II| = 0(1)\,(\mu\to\infty)$ because of (9) (observe that $\varepsilon_n(A,1)=1$).

Remarks. Let $A=C_\alpha$, $B=C_\beta$, then the assumptions of Theorem 2 are satisfied if $0\leqq\alpha,\ \beta\leqq 1$. It follows from $(BA')_{n\mu} = A_{n-\mu}^{\beta-\alpha-1}\dfrac{\mu A_\mu^\alpha}{n A_n^\beta}\,(1\leqq\mu\leqq n)$, $(BA')_{no}=\delta_{no}$ that (8) holds if we assume that $\varepsilon_n(A,b) = 0\,(n^{\beta-\alpha})$. Furthermore, (9) is true for $\beta>0$, but not for $\beta=0$.

THEOREM 3. *Let A be normal and absolutely regular, and let B be triangular and absolutely regular. Furthermore, assume that $A'\,\dot{\leqq}\,0$, $a_n>0$, $0\leqq b_{n\nu}\uparrow$ for $\nu\uparrow(\nu\leqq n)$, and that $b_{n\nu}=\theta_{n\nu}\,a_{n\nu}$, $0\leqq\theta_{n\nu}\,(a_{\rho n}-a_{\rho\nu})\downarrow$ for $\nu\uparrow$ $(\nu\leqq n\leqq\rho)$. Then $\Sigma a_\nu\,\varepsilon_\nu\,(A,b)\in|B|$ whenever $\Sigma a_\nu\in|A|$ if*

$$(10)\qquad \sum_{n=\mu}^{\infty}|\varepsilon_n(A,b)\,(BA')_{n\mu}| = O(1)\qquad(\mu\to\infty)\,.$$

Proof. We have to show that (3) holds, and we write $\varepsilon_\nu=\varepsilon_\nu(A,b)$. Let

$$\sum_{\nu=\mu}^{n} b_{n\nu}\, a'_{\nu\mu}\,\varepsilon_\nu = \varepsilon_n\,(BA')_{n\mu} + \sum_{\nu=\mu}^{n} b_{n\nu}\, a'_{\nu\mu}\,(\varepsilon_\nu-\varepsilon_n) = I + II.$$

Here $\sum_{n=\mu}^{\infty}|I| = O(1)$ because of (10). We have

$$II = \sum_{\rho=\mu}^{n-1} b_\rho \sum_{\nu=\mu}^{\rho} b_{n\nu}\, a_{\rho\nu}\, a'_{\nu\mu} - \sum_{\rho=n}^{\infty} b_\rho\sum_{\nu=\mu}^{n} b_{n\nu}\, a'_{\nu\mu}\,(a_{\rho n}-a_{\rho\nu})\,.$$

It follows as in the proof of Theorem 2 that $\sum_{\nu=\mu}^{\rho} b_{n\nu}\, a_{\rho\nu}\, a'_{\nu\mu}\leqq 0$ for $\rho>\mu$. If we write $c_{\rho n\nu} = \theta_{n\nu}(a_{\rho n} - a_{\rho\nu})$, then

$$\sum_{\nu=\mu}^{n} b_{n\nu} a'_{\nu\mu} (a_{\rho n} - a_{\rho\nu}) = \sum_{\nu=\mu}^{n} (c_{\rho n\nu} - c_{\rho n\mu}) a_{n\nu} a'_{\nu\mu} + c_{\rho n\mu} \delta_{n\mu} \geqq 0 ,$$

and this leads to the estimate

$$K^{-1} |II| \leqq -\Big(\sum_{\rho=\mu}^{n-1} \sum_{\nu=\mu}^{\rho} b_{n\nu} a_{\rho\nu} a'_{\nu\mu} - \sum_{\rho=n}^{\infty} \sum_{\nu=\mu}^{n} b_{n\nu} a'_{\nu\mu} (a_{\rho n} - a_{\rho\nu})\Big) + 2b_{n\mu}$$

$$= - \sum_{\nu=\mu}^{n} b_{n\nu} a'_{\nu\mu} \big(\varepsilon_\nu (A, 1) - \varepsilon_n (A, 1)\big) + 2b_{n\mu}$$

which implies $\sum_{n=\mu}^{\infty} |II| = O(1)$.

Remarks. 1. If $A = B$, then the assumptions of Theorem 3 on A and B are satisfied if $A' \dot{\leqq} 0$, $a_n > 0$, $0 \leqq a_{n\nu} \uparrow$ for $\nu \uparrow (\nu \leqq n)$. In this case (10) is true because of $\varepsilon_n (A, b) = O(1)$.

2. If $B = I$, then the assumptions of Theorem 3 on A and B are satisfied if $A' \dot{\leqq} 0$, $a_n > 0$. In this case (10) reduces to the condition $\sum_{n=\mu}^{\infty} |a'_{n\mu} \varepsilon_n (A, b)| = O(1)$ $(\mu \to \infty)$.

3. If $A = C_\alpha$, $B = C_\beta$, then the assumptions on A and B of Theorem 3 are satisfied for $0 \leqq \alpha$, $\beta \leqq 1$. The condition

$$\theta_{n\nu} (a_{\rho n} - a_{\rho\nu}) \downarrow \text{for } \nu \uparrow (\theta_{n0} = A_n^{\beta-1} A_n^{\alpha} / A_n^{\beta} A_n^{\alpha-1})$$

reduces to the inequality

$$(n - \nu + \beta - 1)(A_{\rho-n}^{\alpha-1} n - A_{\rho-\nu}^{\alpha-1} \nu) \geqq (n - \nu + \alpha - 1)\big(A_{\rho-n}^{\alpha-1} n - A_{\rho-(\nu+1)}^{\alpha-1} (\nu+1)\big)$$

$(\nu < n \leqq \rho)$, and it is sufficient to prove this inequality for $\beta = 0$, i.e. to prove that

$$(11) \qquad \alpha n A_{\rho-n}^{\alpha-1} \leqq (n - \nu + \alpha - 1) A_{\rho-(\nu+1)}^{\alpha-1} (\nu+1) - (n - \nu - 1) A_{\rho-\nu}^{\alpha-1} \nu$$

holds. A short calculation shows that for $\alpha > 0$

$$(\rho - \nu)\left((n - \nu + \alpha - 1)(\nu + 1) - (n - \nu - 1)\nu \frac{A_{\rho-\nu}^{\alpha-1}}{A_{\rho-(\nu+1)}^{\alpha-1}}\right)$$

$$= \alpha\big(\rho + \nu(\rho - n)\big) + (n - \nu - 1)\rho ,$$

and that

$$\frac{A^{\alpha-1}_{\rho-\nu-1}}{A^{\alpha-1}_{\rho-n}} = \frac{(\rho-n)\,!}{(\rho-\nu-1)\,!}\,\frac{1}{(\rho-n+\alpha-1)\ldots(\rho-\nu+\alpha-1)}\,.$$

Using these identities, (11) turns out to be

$$n \leqq \left((\rho+\nu(\rho-n)) + \right.$$

$$\left. + \frac{(n-\nu-1)\,\rho}{\alpha}\right)\frac{(\rho\,-\,n)!}{(\rho-\nu)!}\,\frac{}{(\rho-n+\alpha-1)\ldots(\rho-\nu+\alpha-1)}\,,$$

and this inequality is true if it holds for $\alpha = 1$. But (11) is obviously true for $\alpha = 1$.

If $\varepsilon_n(A, b) = O(n^{\beta-\alpha})$, then (10) is satisfied since (8) is satisfied.

Finally we will discuss the question on conditions which guarantee that (8) or (10) is satisfied if $\varepsilon_n(A, b)$ satisfies (5).

LEMMA 1. *Let* C *be normal,* $C' \stackrel{\bullet}{\leqq} 0$, $c_n > 0$.

(i) *If* $\sum\limits_{n=\nu}^{\infty} c_{n\nu} = 1\ (\nu \geqq 0)$, *then* $\sum\limits_{n=\nu}^{\infty} c'_{n\nu} = 1\ (\nu \geqq 0)$.

(ii) *If* $\sum\limits_{n=\nu}^{\infty} c'_{n\nu} = 1\ (\nu \geqq 0)$, *then* $\sum\limits_{n=\nu}^{\infty} c_{n\nu} \leqq 1\ (\nu \geqq 0)$.

Proof. (i) It follows from $\delta_{n\nu} - c_{n\nu}\, c'_{\nu\nu} = \sum\limits_{\mu=\nu+1}^{n} c_{n\mu}\, c'_{\mu\nu}\ (n \geqq \nu)$ and by summation with respect to n (observe that all terms in the sum on the right are negative) that

$$1 - c'_{\nu\nu} \sum_{n=\nu}^{\infty} c_{n\nu} = \sum_{\mu=\nu+1}^{\infty} c'_{\mu\nu} \sum_{n=\mu}^{\infty} c_{n\mu}\,,$$

and this proves (i).

(ii) It follows from $\delta_{n\nu} = \sum\limits_{\mu=\nu}^{n} c'_{n\mu}\, c_{\mu\nu}\ (\nu \leqq n)$ that for $k \geqq n$

$$1 = \sum_{n=\nu}^{k} \sum_{\mu=\nu}^{n} c'_{n\mu}\, c_{\mu\nu} = \sum_{\mu=\nu}^{k} c_{\mu\nu} \sum_{n=\mu}^{k} c'_{n\mu} \geqq \sum_{\mu=\nu}^{k} c_{\mu\nu}\,,$$

and this proves (ii).

LEMMA 2. *Let A be normal, B be triangular, and assume that* $A' \dot{\leqq} 0$, $a_n > 0$, $b_{n\nu} = \theta_{n\nu}\, a_{n\nu}$.

(i) *If* $0 \leqq \theta_{n\nu} \uparrow$ *for* $\nu \uparrow$ $(\nu \leqq n)$, *then* $BA' \dot{\leqq} 0$.

(ii) *If* $0 \leqq \theta_{n\nu} \downarrow$ *for* $\nu \uparrow$ $(\nu \leqq n)$, *then* $BA' \dot{\geqq} 0$.

Proof. This follows from

$$(BA')_{n\nu} = \sum_{\mu=\nu}^{n} b_{n\mu}\, a'_{\mu\nu} = \sum_{\mu=\nu}^{n} (\theta_{n\mu} - \theta_{n\nu})\, a_{n\mu}\, a'_{\mu\nu} + \theta_{n\nu}\, \delta_{n\nu}\,.$$

Let A and B satisfy the assumptions of Theorem 2, and assume that $b_{n\nu} = \theta_{n\nu}\, a_{n\nu}$, where $\theta_{n\nu} \uparrow$ for $\nu \uparrow$ $(\nu \leqq n)$. It follows from Lemma 1 that $\sum_{n=\mu}^{\infty} a'_{n\mu} = 1$, and it follows from Lemma 2 that $BA' \dot{\leqq} 0$. We have

$$(12) \qquad \sum_{n=\mu}^{\infty} (BA')_{n\mu} = \sum_{n=\mu}^{\infty} \sum_{\nu=\mu}^{n} b_{n\nu}\, a'_{\nu\mu} = \sum_{\nu=\mu}^{\infty} a'_{\nu\mu} \sum_{n=\nu}^{\infty} b_{n\nu} = 1\,,$$

and this implies that

$$(13) \qquad \sum_{n=\mu}^{\infty} |(BA')_{n\mu}| = 2\,(BA')_{\mu\mu} - 1 = 2\,\frac{b_{nn}}{a_{nn}} - 1 = O\left(\frac{b_{nn}}{a_{nn}}\right).$$

If $\varepsilon_n\,(A, b)$ satisfies (5), then it follows from (13) that (8) is satisfied. If $b_{nn}/a_{nn}\downarrow$, then it also follows that (10) holds.

Lemmas 1 and 2 can also be used to prove the following comparison theorem.

THEOREM 4. *Let A and B be normal and absolutely regular. Furthermore, assume that* $A' \dot{\leqq} 0$, $a_n > 0$, $b_{n\nu} = \theta_{n\nu}\, a_{n\nu}$.

(i) *If* $0 \leqq \theta_{n\nu} \uparrow$ *for* $\nu \uparrow$ $(\nu \leqq n)$, *then* $|B| \subseteq |A|$.

(ii) *If* $0 \leqq \theta_{n\nu} \downarrow$ *for* $\nu \downarrow$ $(\nu \leqq n)$, *then* $|B| \supseteq |A|$.

Proof. (i) It follows from Lemma 2 that $BA' = C' \dot{\leqq} 0$, which implies that $A = CB$, $C \dot{\geqq} 0$. It follows from (12) that $\sum_{n=\mu}^{\infty} c'_{n\mu} = 1$, hence we have from Lemma 1 the relation $\sum_{n=\mu}^{\infty} c_{n\mu} \leqq 1$. But $1 = \sum_{n=\mu}^{\infty} a_{n\mu} = \sum_{n=\mu}^{\infty} (CB)_{n\mu} =$

$= \sum_{\nu=\mu}^{\infty} b_{\nu\mu} \sum_{n=\nu}^{\infty} c_{n\nu}$, and it would follow from $\sum_{n=\nu_0}^{\infty} c_{n\nu_0} < 1$ that $\sum_{\nu=\nu_0}^{\infty} b_{\nu\nu_0} \sum_{n=\nu}^{\infty} c_{n\nu} < \sum_{\nu=\nu_0}^{\infty} b_{\nu\nu_0} = 1$, which proves that $\sum_{n=\nu}^{\infty} c_{n\nu} = 1$ for $\nu \geqq 0$. Consequently, C is absolutely regular and, therefore, $|B| \subseteq |A|$.

(ii) It follows from Lemma 2 and (12) that $BA' = C \overset{\bullet}{\geqq} 0$ and $\sum_{n=\mu}^{\infty} c_{n\mu} = 1$ which proves that $|B| \supseteq |A|$ since $B = CA$, where C is absolutely regular.

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(Reçu le 16 septembre 1968)

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