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for some $\lambda_0 > 1$. Accordingly, if the conditions (6.6) and (6.9) hold then (6.4) implies (6.6.) as well as the dominated variation of $1 - F$ and $1 - G$.

Our results permit various paraphrases of the sufficient conditions, and also of the ratio limit theorem itself. That (6.6) by itself is not sufficient is shown by (6.3b); without (6.9) certain subsequences may exhibit the pattern of slow variation, and the conclusion (6.5) must be replaced by a weaker conclusion of the form (6.3b).

7. ON THE TAILS OF INFINITELY DIVISIBLE DISTRIBUTIONS

To illustrate the usefulness of the notion of dominated variation in probabilistic contexts we prove the following

PROPOSITION. *Let H stand for an infinitely divisible probability distribution with Lévy measure $M \{dx\}$. If M varies dominatedly at $+\infty$ then*

$$(7.1) \quad 1 - H(x) \sim M \{ (x, \infty) \}, \quad x \rightarrow +\infty$$

in the sense that the ratio of the two sides tends to unity at all points of continuity. (A very special case involving regular variation is mentioned in [1], p. 540.)

PROOF. We shall show that the general proposition follows easily from the special case where M is supported by the positive half axis and has a finite mass μ . In this case

$$(7.2) \quad M \{ (x, \infty) \} = \mu [1 - F(x)] \quad x > 0$$

where F is a probability distribution on $(0, x)$, and H reduces to the compound Poisson distribution given by

$$(7.3) \quad H(x) = e^{-\mu} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} F^{n*}(x) \quad x > 0.$$

We proceed to prove the assertion (7.1) for distributions of this form assuming that $1 - F$ varies dominatedly. Note that F^{n*} is the distribution of the sum $S_n = X_1 + \dots + X_n$ of n mutually independent random variables with the common distribution F . Since these variables are positive, the event $\{S_n > x\}$ occurs whenever at least one among the n variables exceeds x , and so

$$(7.4) \quad 1 - F^{n*}(x) \geq n [1 - F(x)] - \binom{n}{2} [1 - F(x)]^2$$

by an easily verified inequality named after Bonferoni. Substituting into (7.3) it follows that

$$(7.5) \quad 1 - H(x) \geq \mu [1 - F(x)] - \frac{1}{2} \mu^2 [1 - F(x)]^2 = \\ = M \{ (x, \infty) \} [1 + o(1)], \quad x \rightarrow \infty .$$

To obtain an appraisal in the opposite direction choose $0 < \varepsilon < \frac{1}{2}$ and note that the event $\{S_n > x\}$ cannot occur unless either at least one among the variables X_1, \dots, X_n exceeds $(1 - \varepsilon)x$, or at least two among them exceed $\varepsilon x/n$. Thus

$$(7.6) \quad 1 - F^{n*}(x) \leq n [1 - F((1 - \varepsilon)x)] + \binom{n}{2} [1 - F(\varepsilon x/n)]^2 .$$

To apply the argument used in (7.5) we would have to know that the ratio of the two brackets on the right tends to 0 as $x \rightarrow \infty$. Because of the assumed dominated variation of $1 - F$ this is true for every fixed n , but to make the ratio $< \delta$ we must have $\varepsilon x/n$ sufficiently large, that is, $n \leq ax$, where a is an appropriate constant. On the other hand, if r is the smallest integer exceeding ax and if $ax > 2\mu$ we have trivially

$$(7.7) \quad \sum_{n=r}^{\infty} \frac{\mu^n}{n!} < 2 \left(\frac{\mu}{r} \right)^r < 2 \left(\frac{\mu}{ax} \right)^{ax}$$

and the right side tends to zero faster than any power of x^{-1} . In view of the dominated variation of $1 - F$ this implies that the quantity (7.7) is $o(1 - F(x))$, and this together with (7.6) shows as in (7.5) that

$$(7.8) \quad 1 - H(x) \leq \mu [1 - F(x)](1 + o(1)) .$$

This proves the assertion for distributions of the form (7.3).

For the general case we represent the Lévy measure M as a sum of three measures supported by the intervals $(1, \infty)$, $[-1, 1)$, and $(-\infty, 1]$, respectively. This puts H in the form of a triple convolution, and so we may conceive of H as of the distribution of a sum $X + Y + Z + \text{const.}$ of three infinitely divisible mutually independent random variables such that $X \geq 0$, $Z \leq 0$, and Y has a Lévy measure supported by $[-1, 1]$. It follows that Y has moments of all orders, and hence for arbitrary $\varepsilon > 0$ and n

$$(7.9) \quad P \{ |Y| > \varepsilon x \} = o(x^{-n}) \quad x \rightarrow \infty .$$

Because of the assumed dominated variation $M\{(x, \infty)\}$ decreases more slowly than a certain power $x^{-\alpha}$, and hence the quantity (7.9) is $o(M\{(ax, \infty)\})$ for any fixed $a > 0$. Since $Z \leq 0$ and X has a distribution of the form (7.3) we conclude that

$$(7.10) \quad P\{X + Y + Z > x\} \leq P\{X > (1 - \varepsilon)x\} + P\{Y > \varepsilon x\} \sim \\ \sim M\{((1 - \varepsilon)x, \infty)\}.$$

On the other hand,

$$(7.11) \quad P\{X + Y + Z > x\} \geq P\{X > (1 + \varepsilon)x\} \cdot P\{Y + Z > -\varepsilon x\},$$

and the last factor tends to 1 as $x \rightarrow \infty$. The probabilities on the left are therefore $\sim M\{(x, \infty)\}$, as asserted.

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