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for some $\lambda_0 > 1$. Accordingly, *if the conditions (6.6) and (6.9) hold then (6.4) implies (6.6.) as well as the dominated variation of* $1-F$ *and* $1-G$.

Our results permit various paraphrases of the sufficient conditions, and also of the ratio limit theorem itself. That (6.6) by itself is not sufficient is shown by (6.3*b*); without (6.9) certain subsequences may exhibit the pattern of slow variation, and the conclusion (6.5) must be replaced by a weaker conclusion of the form (6.3*b*).

7. On the tails of infinitely divisible distributions

To illustrate the usefulness of the notion of dominated variation in probabilistic contexts we prove the following

PROPOSITION. *Let H stand for an infinitely divisible probability distribution with Lévy measure* $M\{\mathrm{dx}\}$. *If M varies dominatedly at* $+\infty$ *then*

$$1 - H(x) \sim M\{(x, \infty)\}, \qquad x \to +\infty \tag{7.1}$$

in the sense that the ratio of the two sides tends to unity at all points of continuity. (A very special case involving regular variation is mentioned in [1], p. 540.)

PROOF. We shall show that the general proposition follows easily from the special case where M is supported by the positive half axis and has a finite mass μ. In this case

$$M\{(x, \infty)\} = \mu[1-F(x)] \qquad x > 0 \tag{7.2}$$

where F is a probability distribution on $(0, x)$, and H reduces to the compound Poisson distribution given by

$$H(x) = e^{-\mu} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} F^{n*}(x) \qquad x > 0\,. \tag{7.3}$$

We proceed to prove the assertion (7.1) for distributions of this form assuming that $1-F$ varies dominatedly. Note that F^{n*} is the distribution of the sum $S_n = X_1 + \ldots + X_n$ of n mutually independent random variables with the common distribution F. Since these variables are positive, the event $\{S_n > x\}$ occurs whenever at least one among the n variables exceeds x, and so

$$1 - F^{n*}(x) \geqslant n[1-F(x)] - \binom{n}{2}[1-F(x)]^2 \tag{7.4}$$

by an easily verified inequality named after Bonferoni. Substituting into (7.3) it follows that

$$(7.5) \qquad 1 - H(x) \geqslant \mu[1-F(x)] - \frac{1}{2}\mu^2[1-F(x)]^2 =$$

$$= M\{(x, \infty)\}[1+o(1)], \qquad x \to \infty .$$

To obtain an appraisal in the opposite direction choose $0<\varepsilon<\frac{1}{2}$ and note that the event $\{S_n>x\}$ cannot occur unless either at least one among the variables $X_1, \ldots, X_n$ exceeds $(1-\varepsilon)\,x$, or at least two among them exceed $\varepsilon x/n$. Thus

$$(7.6) \qquad 1 - F^{n^*}(x) \leqslant n[1-F((1-\varepsilon)x)] + \binom{n}{2}[1-F(\varepsilon x/n)]^2 .$$

To apply the argument used in (7.5) we would have to know that the ratio of the two brackets on the right tends to 0 as $x\to\infty$. Because of the assumed dominated variation of $1-F$ this is true for every fixed n, but to make the ratio $<\delta$ we must have $\varepsilon x/n$ sufficiently large, that is, $n\leqslant ax$, where a is an appropriate constant. On the other hand, if r is the smallest integer exceeding ax and if $ax>2\mu$ we have trivially

$$(7.7) \qquad \sum_{n=r}^{\infty} \frac{\mu^n}{n!} < 2\left(\frac{\mu}{r}\right)^r < 2\left(\frac{\mu}{ax}\right)^{ax}$$

and the right side tends to zero faster than any power of x^{-1}. In view of the dominated variation of 1—F this implies that the quantity (7.7) is $o\,(1-F(x))$, and this together with (7.6) shows as in (7.5) that

$$(7.8) \qquad 1 - H(x) \leqslant \mu[1-F(x)](1+o(1)) .$$

This proves the assertion for distributions of the form (7.3).

For the general case we represent the Lévy measure M as a sum of three measures supported by the intervals $(1, \infty)$, $[-1, 1)$, and $(-\infty, 1]$, respectively. This puts H in the form of a triple convolution, and so we may conceive of H as of the distribution of a sum $X+Y+Z+$const. of three infinitely divisible mutually independent random variables such that $X\geqslant 0$, $Z\leqslant 0$, and Y has a Lévy measure supported by $[-1, 1]$. It follows that Y has moments of all orders, and hence for arbitrary $\varepsilon>0$ and n

$$(7.9) \qquad P\{|Y| > \varepsilon x\} = o\,(x^n) \qquad x \to \infty .$$

Because of the assumed dominated variation $M\{(x, \infty)\}$ decreases more slowly than a certain power $x^{-\alpha}$, and hence the quantity (7.9) is $o\,(\mathrm{M}\,\{(ax,\infty)\}$ for any fixed $a>0$. Since $Z\leqslant 0$ and X has a distribution of the form (7.3) we conclude that

$$(7.10)\qquad P\{X+Y+Z>x\}\leqslant P\{X>(1-\varepsilon)x\}+P\{Y>\varepsilon x\}\sim$$
$$\sim M\{((1-\varepsilon)x,\infty)\}.$$

On the other hand,

$$(7.11)\qquad P\{X+Y+Z>x\}\geqslant P\{X>(1+\varepsilon)x\}\cdot P\{Y+Z>-\varepsilon x\},$$

and the last factor tends to 1 as $x\to\infty$. The probabilities on the left are therefore $\sim M\{(x,\infty)\}$, as asserted.

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