

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 15 (1969)  
**Heft:** 1: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** ONE-SIDED ANALOGUES OF KARAMATA's REGULAR VARIATION  
**Autor:** Feller, William  
**Kapitel:** 5. Ratio limit theorems  
**DOI:** <https://doi.org/10.5169/seals-43209>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 24.02.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

Again, if it is known that  $R_U$  is bounded away from 0 then (4.5) shows that (4.2) implies (4.1).

We have thus proved the

COROLLARY. *If  $U$  is of dominated variation with exponent  $\gamma < p$  then (4.1) implies (4.2). Similarly, if  $U_p$  is of dominated variation with exponent  $-q$  where  $q < p$ , then (4.2) entails (4.1). (In each case both functions are of dominated variation.)*

### 5. RATIO LIMIT THEOREMS

Let  $U$  and  $V$  be non-decreasing unbounded functions, and suppose that  $L$  is slowly varying (= regularly varying with exponent 0).

DEFINITION. *We shall say that  $U$  and  $V$  are  $L$ -equivalent and write*

$$(5.1) \quad V \leftrightarrow UL$$

*if the ratio  $UL/V$  tends to 1 at all points of continuity.*

More precisely, it is required that for each  $\varepsilon > 0$  and fixed  $\lambda > 1$

$$(5.2) \quad (1 - \varepsilon) L(t) U(t/\lambda) \leq V(t) \leq (1 + \varepsilon) L(t) U(t\lambda)$$

for all  $t$  sufficiently large.

THEOREM 4. *Let  $U$  be of dominated variation. In order that there exist a slowly varying function  $L$  such that (5.1) holds it is necessary and sufficient that*

$$(5.3) \quad R_U(t) - R_V(t) \rightarrow 0 \quad \text{boundedly.}$$

Needless to say,  $R_V$  and  $\mathcal{J}_V$  are defined by analogy with  $R_U$  in (1.5) and  $\mathcal{J}_U$  in (3.2).

PROOF. (a) *Necessity.* Assume (5.1) and suppose that  $U$  satisfies the basic inequality (2.2). Obviously the slow variation of  $L$  implies that for  $t$  sufficiently large and all  $x > 1$

$$(5.4) \quad \frac{V(tx)}{V(t)} < C' x^{\gamma'}$$

for any pair of constants  $C' > C$  and  $\gamma' > \gamma$ . Thus  $V$  is of dominated variation, and since  $p > \gamma$  the function  $V_p$  exists.

Let  $t_n \rightarrow \infty$  in such a way that the measures associated with  $U(t_n \cdot)/U(t_n)$  tend (in finite intervals) to a limit measure  $m$ . The relation (5.1) implies obviously that the measures associated with  $V(t_n \cdot)/V(t_n)$  tend to the same limit  $m$ . Thus when  $t$  runs through  $\{t_n\}$  we have for fixed  $x > 1$

$$(5.5) \quad \frac{U_p(t) - U_p(tx)}{U(t)t^{-p}} = \int_1^x y^{-p} \frac{U(tdy)}{U(t)} \rightarrow \int_1^x y^{-p} m(dy),$$

and the same relation holds with  $U$  replaced by  $V$ . But (5.4) implies that this passage to the limit is uniform as  $x \rightarrow \infty$ ; it remains valid also for  $x = \infty$  with the right side being finite. We have thus shown that  $R_U(t_n) - R_V(t_n) \rightarrow 0$ . But the  $t_n$  may be picked as elements of an arbitrarily prescribed sequence, and so the limit relation in (5.3) holds pointwise for an arbitrary approach  $t \rightarrow \infty$ . Now we know that the dominated variation of  $U$  and  $V$  implies the boundedness of both  $R_U$  and  $R_V$ , and the condition (5.3) holds true.

(b) *Sufficiency.* The variation of  $U$  being dominated,  $R_U$  remains bounded and so (5.3) implies the boundedness of  $R_V$  and hence the dominated variation of  $V$ . The calculation of part (ii) in section 3 show that

$$(5.6) \quad \frac{s^{-p-1} U(s)}{\mathcal{I}_U(s)} - \frac{s^{-p-1} V(s)}{\mathcal{I}_V(s)} = \frac{p}{t} \left[ \frac{1}{1 + R_U(s)} - \frac{1}{1 + R_V(s)} \right].$$

The expression within brackets is in absolute value bounded by  $|R_U(s) - R_V(s)|$ , and therefore tends to 0 boundedly. Integrating between  $t$  and  $tx > t$  we conclude therefore that

$$(5.7) \quad \log \frac{\mathcal{I}_U(t)}{\mathcal{I}_U(tx)} \cdot \frac{\mathcal{I}_V(tx)}{\mathcal{I}_V(t)} \rightarrow 0.$$

In other words, the ratio  $\mathcal{I}_U/\mathcal{I}_V$  varies slowly, and therefore we can put

$$(5.8) \quad \mathcal{I}_V(t) = L(t) \mathcal{I}_U(t)$$

where  $L$  varies slowly.

We now recall the inequality (3.14) which implies that to each  $\lambda > 1$  there exists an  $\eta < 1$  such that

$$(5.9) \quad \mathcal{I}_U(\lambda t) < \eta \mathcal{I}_U(t)$$

for all  $t$  sufficiently large. From (5.8) we conclude therefore that

$$\begin{aligned}
 (5.10) \quad & \lim \frac{\mathcal{J}_V(t) - \mathcal{J}_V(\lambda t)}{[\mathcal{J}_U(t) - \mathcal{J}_U(\lambda t)] L(t)} = \\
 & = \lim \frac{L(t) \mathcal{J}_U(t) - L(\lambda t) \mathcal{J}_U(\lambda t)}{L(t) \mathcal{J}_U(t) - L(\lambda t) \mathcal{J}_U(\lambda t)} = 1.
 \end{aligned}$$

But the fraction on the left lies between

$$\frac{V(\lambda t)}{U(t) L(t)} \quad \text{and} \quad \frac{V(t)}{U(\lambda t) L(t)}$$

and so (5.1) is true.

## 6. APPLICATION TO TAUBERIAN THEOREMS

If the measure  $U$  varies regularly at infinity, then its Laplace transform  $\omega$  varies regularly at the origin. More precisely, Karamata's now classical Tauberian theorem states that for any  $\alpha \geq 0$  and slowly varying function  $L$  the two relations

$$(6.1) \quad U(x) \sim x^\alpha L(x) \quad \omega(\lambda) \sim \Gamma(\alpha + 1) \lambda^{-\alpha} L(\lambda^{-1})$$

imply each other; here  $x \rightarrow \infty$  but  $\lambda \rightarrow 0$ . [The sign  $\sim$  indicates that the ratio of the two sides tends to 1.] For an example of a probabilistic application suppose that

$$(6.2) \quad U(x) = \int_0^x y^p F(dy)$$

is the truncated  $p^{\text{th}}$  moment of a probability distribution  $F$  on the positive half axis. For simplicity let  $p$  stand for a positive integer. Then  $U_p(x) = 1 - F(x)$  and  $\omega = (-1)^p \phi^{(p)}$  where  $\phi$  is the Laplace-Stieltjes transform of  $F$ . If  $\omega$  varies regularly in accordance with (6.1) then Karamata's relation (1.8) implies that

$$(6.3a) \quad 1 - F(x) \sim \frac{\alpha}{p - \alpha} x^{\alpha - p} L(x) \quad \text{when} \quad \alpha < p$$

$$(6.3b) \quad 1 - F(x) = o(x^\alpha L(x)) \quad \text{when} \quad \alpha = p.$$

(Note that necessarily  $0 \leq \alpha \leq p$  because the measure  $F$  is finite.) In other words, the behavior at the origin of the derivatives of the Laplace transform determines the behavior of the tail  $1 - F(x)$ , and vice versa.