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Again, if it is known that R_U is bounded away from 0 then (4.5) shows that (4.2) implies (4.1).

We have thus proved the

COROLLARY. *If U is of dominated variation with exponent $\gamma < p$ then (4.1) implies (4.2). Similarly, if U_p is of dominated variation with exponent $-q$ where $q < p$, then (4.2) entails (4.1).* (In each case both functions are of dominated variation.)

5. RATIO LIMIT THEOREMS

Let U and V be non-decreasing unbounded functions, and suppose that L is slowly varying ($=$ regularly varying with exponent 0).

DEFINITION. *We shall say that U and V are L-equivalent and write*

$$V \leftrightarrow UL \tag{5.1}$$

if the ratio UL/V tends to 1 at all points of continuity.

More precisely, it is required that for each $\varepsilon > 0$ and fixed $\lambda > 1$

$$(1-\varepsilon)\, L(t)\, U(t/\lambda) \leqslant V(t) \leqslant (1+\varepsilon)\, L(t)\, U(t\lambda) \tag{5.2}$$

for all t sufficiently large.

THEOREM 4. *Let U be of dominated variation. In order that there exist a slowly varying function* L *such that (5.1) holds it is necessary and sufficient that*

$$R_U(t) - R_V(t) \to 0 \qquad \textit{boundedly}. \tag{5.3}$$

Needless to say, R_V and $\mathscr{I}_V$ are defined by analogy with R_U in (1.5) and $\mathscr{I}_U$ in (3.2).

PROOF. (a) *Necessity.* Assume (5.1) and suppose that U satisfies the basic inequality (2.2). Obviously the slow variation of L implies that for t sufficiently large and all $x > 1$

$$\frac{V(tx)}{V(t)} < C' x^{\gamma'} \tag{5.4}$$

for any pair of constants $C' > C$ and $\gamma' > \gamma$. Thus V is of dominated variation, and since $p > \gamma$ the function V_p exists.

Let $t_n \to \infty$ in such a way that the measures associated with $U(t_n \cdot)/U(t_n)$ tend (in finite intervals) to a limit measure m. The relation (5.1) implies obviously that the measures associated with $V(t_n \cdot)/V(t_n)$ tend to the same limit m. Thus when t runs through $\{t_n\}$ we have for fixed $x > 1$

$$(5.5) \qquad \frac{U_p(t) - U_p(tx)}{U(t)\, t^{-p}} = \int_1^x y^{-p} \frac{U(t\,dy)}{U(t)} \to \int_1^x y^{-p}\, m(dy),$$

and the same relation holds with U replaced by V. But (5.4) implies that this passage to the limit is uniform as $x \to \infty$; it remains valid also for $x = \infty$ with the right side being finite. We have thus shown that $R_U(t_n) - R_V(t_n) \to 0$. But the t_n may be picked as elements of an arbitrarily prescribed sequence, and so the limit relation in (5.3) holds pointwise for an arbitrary approach $t \to \infty$. Now we know that the dominated variation of U and V implies the boundedness of both R_U and R_V, and the condition (5.3) holds true.

(b) *Sufficiency.* The variation of U being dominated, R_U remains bounded and so (5.3) implies the boundedness of R_V and hence the dominated variation of V. The calculation of part (ii) in section 3 show that

$$(5.6) \qquad \frac{s^{-p-1}\, U(s)}{\mathscr{I}_U(s)} - \frac{s^{-p-1}\, V(s)}{\mathscr{I}_V(s)} = \frac{p}{t}\left[\frac{1}{1 + R_U(s)} - \frac{1}{1 + R_V(s)}\right].$$

The expression within brackets is in absolute value bounded by $|R_U(s) - R_V(s)|$, and therefore tends to 0 boundedly. Integrating between t and $tx > t$ we conclude therefore that

$$(5.7) \qquad \log \frac{\mathscr{I}_U(t)}{\mathscr{I}_U(tx)} \cdot \frac{\mathscr{I}_V(tx)}{\mathscr{I}_U(t)} \to 0\,.$$

In other words, the ratio $\mathscr{I}_U/\mathscr{I}_V$ varies slowly, and therefore we can put

$$(5.8) \qquad \mathscr{I}_V(t) = L(t)\, \mathscr{I}_U(t)$$

where L varies slowly.

We now recall the inequality (3.14) which implies that to each $\lambda > 1$ there exists an $\eta < 1$ such that

$$(5.9) \qquad \mathscr{I}_U(\lambda t) < \eta\, \mathscr{I}_U(t)$$

for all t sufficiently large. From (5.8) we conclude therefore that

$$(5.10) \qquad \lim \frac{\mathcal{I}_V(t) - \mathcal{I}_V(\lambda t)}{[\mathcal{I}_U(t) - \mathcal{I}_U(\lambda t)]\, L(t)} =$$

$$= \lim \frac{L(t)\, \mathcal{I}_U(t) - L(\lambda t)\, \mathcal{I}_U(\lambda t)}{L(t)\, \mathcal{I}_U(t) - L(t)\, \mathcal{I}_U(\lambda t)} = 1\,.$$

But the fraction on the left lies between

$$\frac{V(\lambda t)}{U(t)\, L(t)} \quad \text{and} \quad \frac{V(t)}{U(\lambda t)\, L(t)}$$

and so (5.1) is true.

6. Application to Tauberian theorems

If the measure U varies regularly at infinity, then its Laplace transform ω varies regularly at the origin. More precisely, Karamata's now classical Tauberian theorem states that for any $\alpha \geqslant 0$ and slowly varying function L the two relations

$$(6.1) \qquad U(x) \sim x^{\alpha} L(x) \qquad \omega(\lambda) \sim \Gamma(\alpha+1)\, \lambda^{-\alpha} L(\lambda^{-1})$$

imply each other; here $x \to \infty$ but $\lambda \to 0$. [The sign $\sim$ indicates that the ratio of the two sides tends to 1.] For an example of a probabilistic application suppose that

$$(6.2) \qquad U(x) = \int_0^x y^p F(dy)$$

is the truncated p^{th} moment of a probability distribution F on the positive half axis. For simplicity let p stand for a positive integer. Then $U_p(x) = 1 - F(x)$ and $\omega = (-1)^p \phi^{(p)}$ where ϕ is the Laplace-Stieltjes transform of F. If ω varies regularly in accordance with (6.1) then Karamata's relation (1.8) implies that

$$(6.3a) \qquad 1 - F(x) \sim \frac{\alpha}{p-\alpha}\, x^{\alpha-p} L(x) \qquad \textit{when} \qquad \alpha < p$$

$$(6.3b) \qquad 1 - F(x) = o(x^{\alpha} L(x)) \qquad \textit{when} \qquad \alpha = p\,.$$

(Note that necessarily $0 \leqslant \alpha \leqslant p$ because the measure F is finite.) In other words, the behavior at the origin of the derivatives of the Laplace transform determines the behavior of the tail $1 - F(x)$, and vice versa.